MODERN PLANE GEOMETRY

WELLS AND HART

Mathematics Committee - 1927

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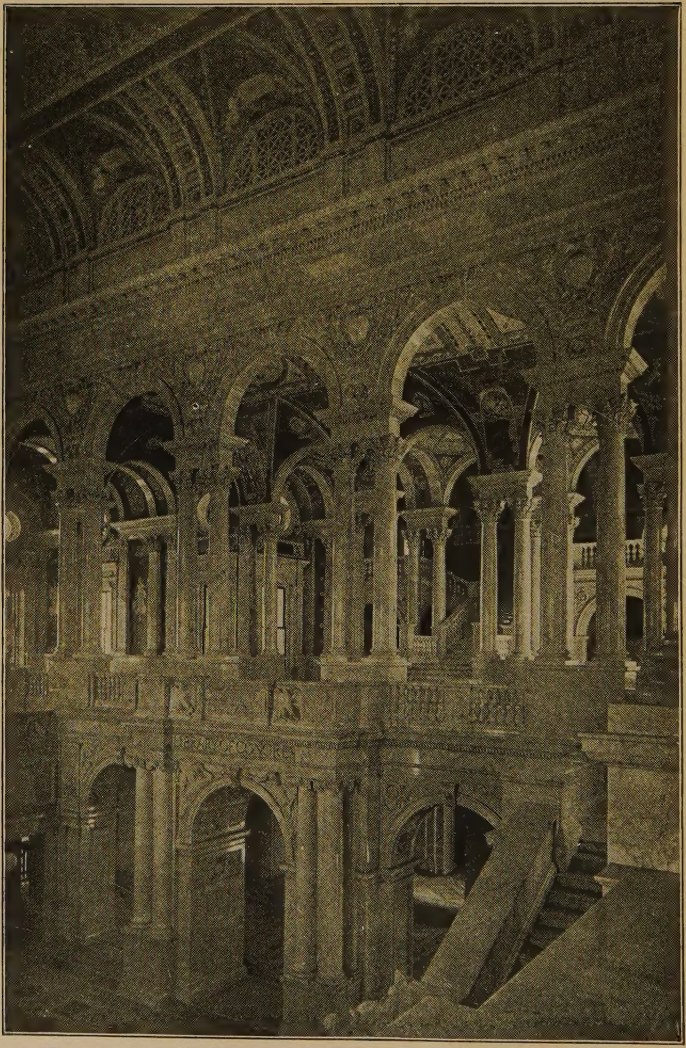

A View in the Congressional Library, Illustrating the Use of Geometry in Architecture

MODERN PLANE GEOMETRY

GRADED COURSE

BY

WEBSTER WELLS, S.B.
AUTHOR OF A SERIES OF TEXTS ON MATHEMATICS

AND

WALTER W. HART, A.B.
ASSOCIATE PROFESSOR OF MATHEMATICS, SCHOOL OF EDUCATION
UNIVERSITY OF WISCONSIN

D. C. HEATH AND COMPANY
BOSTON NEW YORK CHICAGO LONDON
ATLANTA SAN FRANCISCO DALLAS

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PRINTED IN U. S. A.

PREFACE

MODERN PLANE GEOMETRY is the third in a sequence of three geometries. The first was Wells' Essentials of Geometry, which was a forerunner of texts which prevent mere memorizing of the subject. The second was Wells and Hart's Plane Geometry (1915), which carried on the ideals of its predecessor by a wealth of devices, and anticipated by several years most of the National Committee's recommendations pertaining to geometry. The present text is a refinement of the ideals of its predecessors and is the natural result of ten more years of experience in teaching the subject to high school pupils.

This text will enable teachers to adapt instruction in geometry to the capacities of their pupils, which is without doubt the foremost ideal and need of current educational effort. This is accomplished by having in each Book a minimum course and two kinds of optional material, a unique organization of the subject matter designed to facilitate departures from a single fixed course for all pupils.

The minimum course

The minimum course includes all the theorems which are indicated as fundamental by the College Entrance Requirements Board. These theorems are marked by a star (*), as in the Board's list. The other theorems of the minimum course also all appear in the list of the Board; they are necessary to organize a logical sequence. Almost every proposition is accompanied by one or more exercises, which have been selected, because of their simplicity, to aid in teaching the propositions and in developing skill in solving originals. A unique summarizing review comes at the close of the minimum course of each Book. (See pp. 83, 131, 182, 218, 254.) Finally, to complete the minimum course, standard tests similar to these reviews with directions for giving and scoring them, and with tentative standard scores,

have been prepared and can be obtained from the publishers at small cost.

The minimum courses of the five Books are independent of all the optional topics. Together they constitute an exceptionally brief logically developed course in geometry which complies with the requirements of the College Entrance Requirements Board, — and, lest there be misunderstanding, this is in all respects the briefest course which has ever been recommended.

To limit the opportunities of the able pupils of a class to this minimum course is just *as grave a pedagogical error* as to overtax their less capable fellows. The provisions for the more able pupils are as definite as for the weaker ones.

Supplementary exercises

First, there is a *unique system* of cross references to supplementary exercises which are placed at the end of the text. (*E.g.* see Note, at the foot of p. 55.) This arrangement furnishes the busy teacher additional exercises from time to time, for the solution of which, she is assured, no theorem is required which has not yet been introduced in the text. These exercises are a means of extending the minimum course for the benefit of the pupils who can and should do additional work.

Optional topics

Second, optional topics appear at the close of each Book. (See pp. 86, 134, etc.) This material is made optional, (*a*) because it is designated as being subsidiary in the lists of the College Entrance Requirements Board and the National Committee, and (*b*) because the minimum course is quite sufficient for the less capable pupils. These topics all appear in some form in current texts. The *uniqueness of this text* consists in definitely setting them apart as not required in the logical development of the minimum course. As organized in this text, these topics are mutually independent, except as stated in the descriptions of them, so that teachers may select from them at will in order to provide for the wide variations in ability in pupils and in classes.

Besides this distinctive organization of the text in terms of a minimum course, with optional topics, the following characteristics of the text should be mentioned.

The *introduction* is a brief course in intuitive geometry. Concepts, definitions, axioms, etc. which are required immediately, are taught by means of exercises in drawing and measuring. Others are postponed until they are needed in the text. (See pp. 25, 65, 76, etc.) The geometry tool, originated by the author and furnished with this text, is the only tool needed until the first construction with compasses is reached on page 34. Constructions with compasses are neither a necessary nor an easy introduction to the study of geometry.

The undefined terms (§§ 3, 4), the definitions, and the axioms (which all appeared in the text of 1915) are the ones which have been given approval in the report of the National Committee in 1923. (See §§ 15, 16, 20, 21, 49, 120, 136, 320, 322.)

Two *alternative arrangements of a formal demonstration* are illustrated on pages 20 and 21. The parallel arrangement, which in many respects is more convenient for the printed page than for manuscript prepared by pupils, has been used in this text in response to requests from many teachers.

Authorities and details of proofs which pupils can supply are *omitted* from the demonstrations in the text, because it is certainly illogical to give complete demonstrations in the text, and, at the same time, to expect pupils to solve originals whose demonstrations are even more difficult.

The *fundamental constructions* have been placed early in the text. (See pp. 34, 37, 38, etc.) By this order of propositions: the only hypothetical construction is the one used on page 32, the constructions can be proved when they are introduced, and their proofs give continued drill in the use of the congruence theorems and of Fundamental Plan I. (See p. 28.) This arrangement also postpones the subject of parallels, with the attendant necessary indirect proof of at least one theorem.

Systematic training in the solution of originals is accomplished by numerous devices scattered through the text. The elements of this training are: (*a*) omission of parts of demonstrations in the text, to prevent mere memorizing of demonstrations; (*b*) consistent but simple statement of the "Plan" of each demonstration in the text, — a device introduced by the author in the text of 1915; (*c*) formulation of and practice

in the use of a few fundamental plans (see pp. 28, 48, 87, 103, 164, and 170); (*d*) inclusion of illustrative solutions of a few exercises, each of which prevents a definite difficulty of pupils when such solutions are omitted (see pp. 27, 28, 164); (*e*) inclusion of one or more exercises with almost every proposition; (*f*) inclusion of lists of miscellaneous exercises periodically, as well as the supplementary exercises at the end of the book; (*g*) gradation of the exercises, — evidenced by simple one-step exercises on page 27, two-step exercises on page 29, exercises accompanied by the hypothesis, conclusion, and figure, exercises later without these aids, and use in the minimum course only of exercises which experience has shown are easy; (*h*) discussion of methods of proving unproved theorems (pp. 58, 59), of proof (p. 94), of analysis (pp. 94, 143), of attacking a locus exercise (p. 139).

The *treatment of proportion* has been limited to the parts actually needed in geometry. (See pp. 153, 154.) The necessary theorems have been proved informally. For the unsuggestive terms "composition" and "division," the more logical ones "addition" and "subtraction" have been substituted (p. 154).

Attention is called to the *optional topic, numerical trigonometry*, on page 187. Notwithstanding efforts to popularize this topic as a part of ninth-grade algebra, the majority of experienced teachers favor teaching it where it appears in this text. This is the logical and also the expedient place for it. Here, quite as much as in the ninth grade, is it needed to add to the interest and the practical value of the course. Moreover, since the first opportunity to take the examinations of the College Entrance Requirements Board comes at the close of the tenth year, this location is best for the pupil preparing for the examinations. In connection with this topic, observe the use made of it in computing the sides, apothems, and radii of regular polygons (p. 265) as a substitute for the more laborious computations usually given in geometries.

Other simple applications of geometry appear in such exercises as Exs. 108–110, p. 24; Ex. 20, p. 29; Ex. 53, p. 36; Exs. 159–165, p. 64; etc. More difficult and less common applications appear in the optional sections, as on p. 149.

Observe the illustrative solutions of examples from arithmetic and algebra (pp. 311–314). These meet a need which teachers know exists.

The function concept has been introduced in such exercises as Ex. 209, p. 144; Ex. 210, p. 145; Exs. 195, 196, p. 193.

A brief historical introduction appears at the beginning of the text. Thereafter, historical notes have been introduced whenever opportune.

The incommensurable cases of Books II and III have been postponed to the appendix. The corresponding difficulty in Book IV has been avoided by adopting as the definition of area of a rectangle the well-known formula (p. 205), thus following a practice of some foreign and some American texts, and one which is thoroughly scientific. The mensuration of the circle is made informal as to the theory of limits, as suggested by numerous reports.

No effort has been spared to make the book attractive and durable. Important topics and almost all propositions start at the top of a page, and every demonstration is completed without turning to the next page. Authorities, notes, remarks, and exercises are printed in a second size type in order to make the main body of the text more emphatic.

CONTENTS

SYMBOLS AND ABBREVIATIONS

It is not necessary to learn any of these symbols or abbreviations until they are introduced in the text.

Symbols

$\frown$, arc.	$\angle$, angle.
$=$, is equal to; equals.	$\triangle$, triangle.
$>$, is greater than.	$\square$, parallelogram.
$<$, is less than.	$\square$, rectangle.
$\parallel$, is parallel to; parallel.	$\odot$, circle.
$\perp$, is perpendicular to.	$\therefore$, therefore.
$\perp$, perpendicular.	$\equiv$, is identically equal to.
$\sim$, is similar to.	$\to$, approaches as limit.
$\cong$, is congruent to.	

Any symbol representing a noun is converted into the plural by affixing the letter *s*; thus $\angle s$ means angles.

Abbreviations

Adj.,	adjacent.	*Def.*,	definition.
Alt.,	alternate.	*Ex.*,	exercise.
Ax.,	axiom.	*Ext.*,	exterior.
Comp.,	complementary.	*Hyp.*,	hypothesis.
Con.,	conclusion.	*Int.*,	interior.
Cong.,	congruent.	*Rect.*,	rectangle.
Const.,	construction.	*Rt.*,	right.
Cor.,	corollary.	*St.*,	straight.
Corres.,	corresponding.	*Supp.*,	supplementary.

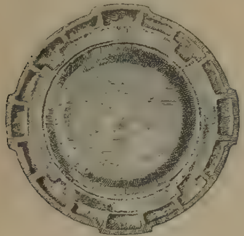
PLANE GEOMETRY

INTRODUCTION

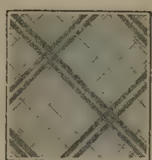
What is geometry? The rectangle, the circle, the cube, straight lines, parallel lines, etc., are geometrical figures. Such figures appear in nature and in the works of man.

A beautiful example of the use of geometrical figures in architecture is the Lobby of The Congressional Library.

Below are examples of the use of geometrical figures in artistic designs.

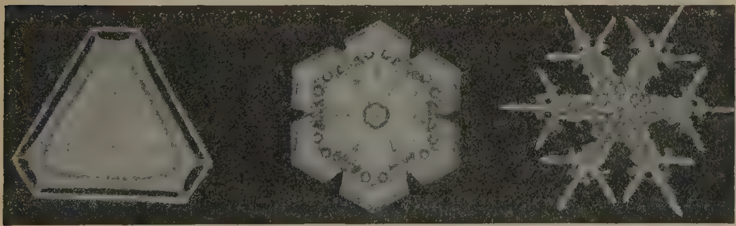


AN ARTISTIC TRAY



GINGHAM PATTERN

ORNAMENTAL IRON WORK



GEOMETRICAL FIGURES FOUND IN SNOWFLAKES

In geometry, you will study such figures, learning how to construct and measure them, and remarkable facts about them.

How geometry has developed. Geometry as it is now studied has been handed down to us from the Greeks. The word *geometry* is derived from two Greek words meaning *the earth* and *to measure;* this fact is evidence that the Greeks believed that geometry was intimately associated with or else had been developed out of the practical need of measuring the earth.

The Greeks received their start in geometry from the Egyptians. Evidence that the Egyptians had a knowledge of practical geometry is found in a papyrus which is now in the British Museum. This papyrus was written sometime between 2000 B.C. and 1700 B.C. by an Egyptian commonly called Ahmes. It, in turn, is a copy of one which had been written still earlier. It contains, among other interesting mathematical records, some formulas for measuring geometrical figures. Some of the formulas are incorrect; the formula for the area of a circle, while remarkably good, is less accurate than that which was given later by Archimedes, a Greek. Their pyramids and other marvelous structures and the geometrical designs found on the walls of their buildings also are evidence of their acquaintance with geometry. Most of their geometry pertained to areas and volumes of figures. They must have obtained their formulas by experiment and by observation. Herodotus, the Greek historian and traveler, is said to be responsible for the story that the Egyptians developed these rules for areas because of the necessity of frequently surveying land which was inundated by floods from the Nile. From the little evidence we have, it seems safe to conclude that Egyptian geometry was meager in extent, crude in its methods, and practical.

Thales of Miletus (640 B.C.–546 B.C.), one of the "seven wise men" of Greece, is given special credit for introducing Egyptian geometry to Greece. He, like other men of wealth and scholars of his time, traveled and studied in Egypt. According to tradition he surprised

his teachers by his ability in geometry. Upon his return to Greece, he taught his friends the geometry he had learned during his travels, and, with them, made further discoveries in geometry.

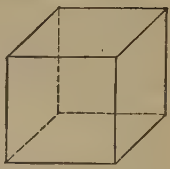
The Greeks became interested in geometry for its own sake as well as for its usefulness. As a result, in the three hundred years following the time of Thales, geometry grew into a great science in their schools, far exceeding the geometry of the Egyptians in the number and interest of the facts discovered and in the accuracy and usefulness of the results.

Pythagoras (580 B.C.–500 B.C.) and Plato (429 B.C.–347 B.C.) were the leaders of two groups of scholars who were responsible for much of the advance made in the subject. Plato's respect for geometry was so great that he had over the entrance to his school the notice "Let no one who is unacquainted with geometry enter here."

During these three hundred years texts were written, but it remained for Euclid to write what became the standard text. Euclid lived between 330 B.C. and 275 B.C. He was one of the first mathematicians who taught at the great University of Alexandria. As a teacher, he felt the need of a text by which to lead beginners through the known facts of elementary geometry. He therefore gathered together and systematized the beginnings of geometry in a series of booklets known as Euclid's Elements. With the exception of the Bible, no other ancient text, probably, has been studied as much as the Elements. It has stood as the model for all subsequent texts on geometry. During the twenty-two hundred years since the time of Euclid, geometry has been studied by all civilized peoples and has been enriched from time to time by their mathematicians. It is now the privilege of, and it is now possible for boys and girls of high school age to learn in one year more geometry than a majority of the wise men of ancient times knew.

PREPARATORY GEOMETRY

1. Plane surface. The adjoining figure is a *cube*. The surfaces, which are smooth and flat, are called *plane surfaces;* they are such that a straightedge (ruler) will touch the surface at all points of the straightedge, no matter where the plane surface may be tested.



2. Plane geometry is the study of figures like the square, the triangle, the circle, etc., — *figures which lie in a plane surface.*

A **plane geometrical figure** is a combination of points and lines which lie in one plane surface. *Only such figures are considered in plane geometry.*

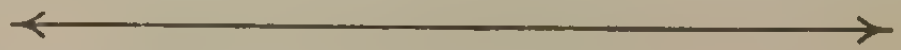
3. A **point** is represented to the eye by a small dot. . *A*

A point is named by placing beside it a capital printed letter; as point *A*. A point represents position only.

4. A **straight line** is represented to the eye by a mark made by drawing a pencil, a pen, or a piece of crayon along the edge of a straightedge.

A straight line extends infinitely far in each of two directions. It does not have any end points.

Since only a part of any straight line can be drawn on a piece of paper, arrow heads will be placed at its ends, *for the present*, to indicate that the line extends infinitely far.



Straight Line

Ex. 1. Place upon paper a single point. Draw through it one straight line. Draw through it another straight line; a third.

Ex. 2. How many straight lines can be drawn through one point?

Ex. 3. (*a*) Place upon paper a point *A* and a point *B*. Draw through *A* and *B* a straight line.

(*b*) Try to draw a second straight line through *A* to *B*.

Ex. 4. How many different straight lines can be drawn through two points?

5. Memorize the following statements:

(*a*) *One and only one straight line can be drawn through two points;*

or *two points determine a straight line.*

(*b*) *A straight line extends infinitely far in each of two directions.*

6. The straight line determined by points A and B is called the *line AB.*

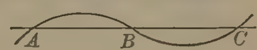
A line may be named also by placing a single small letter beside it; as *line m.*

Ex. 5. Select three points which are not in one straight line. Letter them A, B, and C. Draw and name the different straight lines determined by them taken two at a time.

Ex. 6. If four towns A, B, C, and D are situated so that no three can be connected by one straight road, how many roads must be constructed if each town is to be connected with each of the others by a straight road? Illustrate by a drawing.

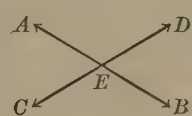
7. A curved line is a line of which no part is straight.

8. Two lines, straight or curved, **intersect** if they have one or more common points. The common points are called **points of intersection.**



9. *Two straight lines can intersect at only one point.*

If they were to intersect in two points, there would be two straight lines through these two points, and this is impossible according to § 5 (*a*).



This fact is also stated thus:

Two intersecting straight lines determine a point.

Ex. 7. Draw three straight lines intersecting by pairs which do not all pass through one point. How many points do they determine?

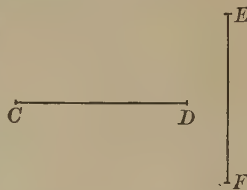
Ex. 8. How many points are determined by four straight lines intersecting by pairs, if no three of them pass through a common point?

10. A **line-segment** or **segment** is the part of a straight line between two points of the line; as, segment RS.

A segment has two end points.

A segment can be extended in each of two directions.

11. Two segments are **equal** if they can be placed so that the ends of the one are exactly upon the ends of the other. We then say that they *coincide.*



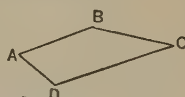
The equality of two segments EF and CD can be tested in three ways.

(*a*) *If you have compasses, separate the points until one point can be placed on C and the other, at the same time, on D. Then, without changing the distance between the points of the compasses, place one point on E and decide whether the other can be made to fall on F.*

(*b*) On transparent paper, take a copy of CD and try to make it coincide with EF. EF is less than ($<$) CD, if EF equals a part of CD.

(*c*) On the straight edge of a piece of paper, take a copy of CD and determine whether the copy can be made to coincide with EF.

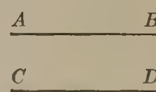
Ex. 9. Determine the relative lengths of AB and BC; of AB and CD; of AB and AD.



Ex. 10. Draw segments AB and CD, with AB greater than CD.

(*a*) On a line of indefinite length, mark off a segment equal to $AB + CD$. (*b*) Mark off a segment equal to $AB - CD$.

Ex. 11. (*a*) Suppose that AB is placed upon CD with point A on point C. Where will B fall if $AB = CD$?



(*b*) Where will B fall if $AB = \frac{1}{2} CD$?

(*c*) Where will B fall if AB is greater than CD?

Ex. 12. Suppose that two segments are each equal to a third segment. How do these two segments compare with each other?

Ex. 13. Suppose that two segments are each equal to equal segments. How do these segments compare with each other?

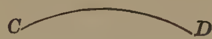
Ex. 14. Complete the following sentences:

(*a*) If equal segments are added to equal segments, the sums are ⋯

(*b*) If equal segments are subtracted from equal segments, the remainders are ⋯

12. A **broken line** is a line composed of connected line-segments.

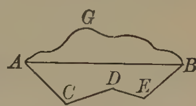
STRAIGHT LINE



CURVED LINE

BROKEN LINE

The word "line" will mean a straight line hereafter unless otherwise specified and "segment" will mean straight line-segment.



13. It will be assumed that *the straight line-segment is the shortest line between two points.*

The **distance** between two points is the length of the segment of the straight line between the points.

To obtain a straight line between two points, a carpenter stretches a piece of twine between the two points. In doing so, he assumes that the shortest line between two points is the straight line.

Ex. 15. Place upon paper points A, B, and C so that they do not all lie upon a straight line. Draw segments AB, BC, and AC.

(*a*) Compare the longest segment with the sum of the other two segments.

(*b*) Compare one segment with the difference of the other two.

14. **A point bisects a segment** if it divides the segment into two equal segments. The point is called the **mid-point** of the segment.

Thus, C bisects AB if $AC = CB$.

It will be assumed that *a segment has one and only one mid-point.*

Ex. 16. Determine as suggested in § 11, *a*, *b*, or *c* whether C does actually bisect AB. If it does, what part of AB is AC? Does D bisect AC? Does E bisect CB? (See Fig. § 14.)

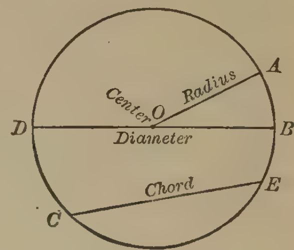
Ex. 17. Draw a segment of any length and locate freehand the point which you think bisects the segment. Test the two parts to determine whether you actually located the mid-point of the segment. (Continue this exercise until you can approximately bisect a segment in this manner.)

Ex. 18. What must be true about halves of equal segments?

15. Lines like the adjoining ones are called **closed lines.**

A closed line incloses a portion of the plane.

16. A **circle** is a closed curved line, lying in a plane, such that all points of it are equidistant from a point within called the **center.**



A **radius** of a circle is the segment from the center to any point on the circle; as OA.

A **diameter** of a circle is a segment drawn through the center of the circle with its ends on the circle; as BD.

A **chord** of a circle is the segment joining any two points of the circle; as, chord CE.

An **arc** ($\frown$) of a circle is the part of the circle between two points of the circle; as, **arc** AB ($\overset{\frown}{AB}$). Since it is possible to go from A to B along the circle in two ways, there are actually two arcs AB. The smaller one is called the **minor arc** AB, and the larger one, the **major arc** AB. Usually, "arc AB" means the minor arc AB.

17. *A circle can be drawn with any point as center and any given segment as radius.*

18. Equal circles are circles having equal radii. They can be made to coincide.

19. *All radii of the same circle or equal circles are equal.*

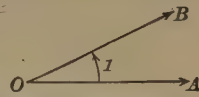
Ex. 19. Draw two circles having the same center with radii of 1.5 in. and 2 in. respectively.

Ex. 20. Draw a circle and a straight line which intersects it. How many points of intersection are there?

Ex. 21. Draw two circles that intersect. How many points of intersection are there?

20. A **half-line** or **ray** is the part of a straight line in one direction from a given point on the line.

21. An **angle** ($\angle$) is the figure formed by two rays drawn from the same point.



This definition was introduced by a mathematician, Bertrand, in 1778.

Since the rays extend infinitely far, the size of the angle does not depend upon the length of its sides.

The common point is called the **vertex** of the angle.

The two rays are called the **sides** of the angle.

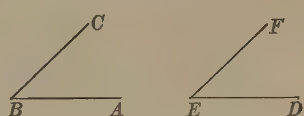
One may imagine a ray starting from the position OA and turning about point O in *counter-clockwise direction* until it occupies the position OB. OA is then called the *initial* line, and OB the *terminal* line. The portion of the plane over which the line would pass is said to be within the angle.

An angle may be named by the letter at its vertex if there is only one angle having that vertex; as, $\angle O$.

An angle may be indicated by a number placed within it near its vertex; as, $\angle 1$.

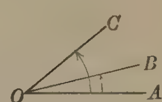
An angle may be named by reading the letters A, O, B; as $\angle AOB$, where the vertex letter O is placed between the other two letters, and the initial line is read first.

22. Two **angles** are **equal** if they can be made to coincide, — that is, fit together.



Thus, if $\angle E$ can be placed upon $\angle B$ so that point E is on point B, line ED on line BA, and line EF on line BC, then $\angle E$ equals $\angle B$.

$\angle AOB$ is less than $\angle AOC$ if it equals a part of $\angle AOC$.



Ex. **22.** How many end-points has a line-segment? A ray? A line?

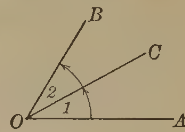
Ex. **23.** Make on transparent paper a tracing of $\angle DEF$ (§ 22). Place the tracing over $\angle ABC$ and determine whether the angles are equal.

Ex. **24.** What must be true about two angles each of which is equal to a third angle?

23. **A line bisects an angle** if it divides the angle into two equal angles.

The line is called the **bisector** of the angle.

Thus, OC bisects $\angle AOB$ if $\angle 1 = \angle 2$.



It will be assumed that *an angle has one and only one bisector.*

Ex. 25. What part of $\angle AOB$ is $\angle 1$? (See Fig. § 23.)

Ex. 26. Draw any angle. Draw a line which you think bisects the angle. Test the equality of the two parts of the angle by means of tracing paper.

(Repeat this exercise until you can approximately bisect an angle in this manner.)

Ex. 27. What must be true about halves of equal angles?

Ex. 28. Suppose that $\angle ABC$ is placed upon $\angle DEF$ so that point B is on point E, and line BA is on line ED.

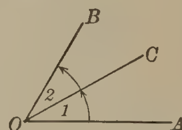
Where will BC fall if $\angle ABC$ is equal to $\angle DEF$?

Where will BC fall if $\angle ABC$ is less than $\angle DEF$?

Where will BC fall if $\angle ABC$ is greater than $\angle DEF$?

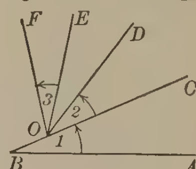
24. **Adjacent angles** are two angles that have a common vertex and a common side between them.

Thus, $\angle 1$ and $\angle 2$ are adjacent angles.

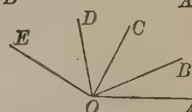


Ex. 29. In the adjoining figure:

(*a*) Is $\angle 1$ adjacent to $\angle 2$? Why?
(*b*) Is $\angle 2$ adjacent to $\angle 3$? Why?



25. **Two adjacent angles are added** thus:



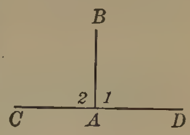
$$\angle AOB + \angle BOC = \angle AOC.$$

Also, $\quad \angle AOC - \angle BOC = \angle AOB.$

Ex. 30. In the figure (§ 25), read the angle which represents:

(*a*) $\angle AOB + \angle BOD$;
(*b*) $\angle BOC + \angle COD$;
(*c*) $\angle BOC + \angle COD + \angle DOE$;
(*d*) $\angle BOE - \angle DOE$.

26. If one straight line meets another straight line so that the adjacent angles formed are equal, each of these angles is a **right angle**; as, $\angle 1$ and $\angle 2$.



27. It will be assumed that *all right angles are equal.*

28. The usual unit for measuring angles is the **angle-degree** or **degree**, which is one ninetieth of a right angle. The degree is divided into sixty equal parts, called minutes, and the minute into sixty equal parts, called seconds.

Degrees, minutes, and seconds are represented by the symbols °, ′, and ″ respectively.

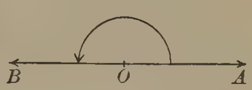
Thus, 42° 22′ 37″ denotes an angle of 42 degrees, 22 minutes, and 37 seconds.

Ex. 31. Point out in your classroom some right angles.

Ex. 32. Fold a piece of paper so as to make a straight line; then fold it again so as to form two equal adjacent angles.

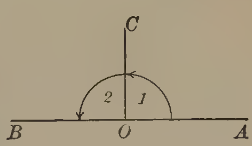
Ex. 33. How many degrees are there in: $\frac{1}{2}$ rt. $\angle$? $\frac{1}{4}$ rt. $\angle$?

29. A **straight angle** is an angle whose sides lie in a straight line on opposite sides of its vertex.



30. If CO meets AB so that $\angle 1$ and $\angle 2$ are equal, each angle is a right angle by § 26; also, $\angle AOB$ is a straight angle by § 29.

A straight angle equals two right angles.



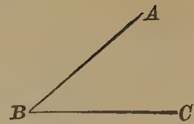
31. Since a straight angle is equal to two right angles (§ 30) and since all right angles are equal (§ 27), it is evident that *all straight angles are equal.*

Ex. 34. How many degrees are there in a straight angle?

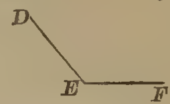
Ex. 35. What part of a straight angle is a right angle?

Ex. 36. At what hour do the hands of a clock form a straight angle?

32. An **acute angle** is an angle which is less than a right angle; as $\angle CBA$.

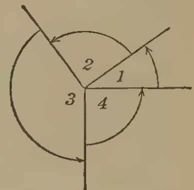


An **obtuse angle** is an angle which is greater than a right angle, and less than a straight angle; as $\angle FED$.



Acute and obtuse angles, collectively, are called **oblique angles.**

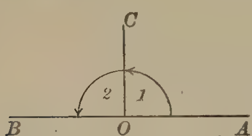
Ex. 37. What kind of angle in the adjoining figure is $\angle 1$? $\angle 2$? $\angle 3$? $\angle 4$?



Ex. 38. What kind of angle do the hands of a clock form at 3 o'clock? At 1 o'clock?

Ex. 39. How many degrees are there in each of the angles in Exercise 38?

33. Two lines are **perpendicular** ($\perp$) if they form a right angle.

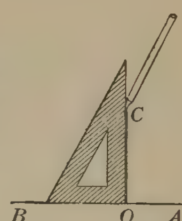


$CO \perp AB$ if $\angle 1$ and $\angle 2$ are right angles.

$AO \perp BO$ if $\angle AOB$ is a right angle.

34. To draw a perpendicular to a line.

A practical method of drawing a perpendicular to a line is to use a pattern right angle as illustrated in the adjoining figure. Draughtsmen use the tool pictured. You can use the geometry tool furnished with this text.



(*a*) *To draw a perpendicular to a line AB at O*, place the card with one edge along AB, and a corner at O. Then draw from O a line along the other edge, passing through O.

(*b*) *To draw a perpendicular to line AB from a point C*, place the card with one edge along AB, and a perpendicular edge passing through C. Then draw a line from C to AB.

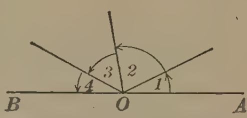
Ex. 40. Draw a segment AB, making it 2 inches long. Place C at its center. At C, draw a perpendicular to AB, above AB. Take any point X on this perpendicular, and connect it with A and B. Measure AX and BX. How do they compare?

Ex. 41. Draw a horizontal segment AB. Place a point C about 2 inches above it, without measuring the distance. From C, draw a perpendicular to AB, meeting it at D. Measure CD.

Ex. 42. Draw a segment AB, about 3 inches long. On it place points X and Y, 2 inches apart. Above AB, at X, draw XR perpendicular to AB; make XR 2 inches long. Above AB, at Y, draw YS perpendicular to AB; make YS 2 inches long. Measure RS.

Ex. 43. Draw a vertical segment CD. On it, place M and N, 2 inches apart. On the right side of CD, at M, draw MA perpendicular to CD; make MA 3 inches long. Also at N, draw NB perpendicular to CD on the right side of CD; make NB 3 inches long. Draw and measure AB.

35. *The sum of all the successive adjacent angles around a point on one side of a straight line through the point is one straight angle, or 180°.*

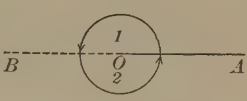


Thus, $\angle 1 + \angle 2 + \angle 3 + \angle 4 = \angle AOB$, and $\angle AOB = 1$ st. $\angle$.

Ex. 44. If $\angle 1 = \angle 2 = \angle 3 = \angle 4$, how many degrees are there in each angle?

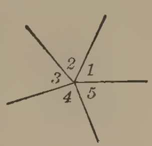
Ex. 45. If, in the figure for § 35, $\angle 2 =$ three times $\angle 1$, $\angle 3 = \angle 2$, and $\angle 4 =$ twice $\angle 1$, how many degrees are there in each angle. (Let $x =$ the No. of degrees in $\angle 1$.)

36. *The sum of all the successive adjacent angles around a point in a plane is two straight angles, or 360 degrees.*



If AO is extended to B, then $\angle 1 = 1$ st. $\angle$, and also $\angle 2$. Therefore $\angle 1 + \angle 2 = 2$ st. $\angle$s.

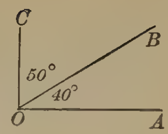
The total angular magnitude around a point is called a **perigon**.



Ex. 46. Through what angle does the minute hand of a clock revolve in one half hour? In one hour?

Ex. 47. How large would each angle of the adjoining figure be if the angles were all equal?

37. Two angles are **complementary** if their sum is equal to a right angle. Each of the angles is called the **complement** of the other.



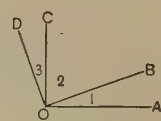
Thus, the complement of 40° is 50°.

Ex. 48. What is the complement of 10°? of 25°? of 50°? of 90°? of 0°? of 45°? of x°?

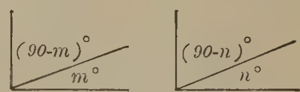
Ex. 49. How large is the angle which equals its complement? (Let x = the no. of degrees in the complement.)

Ex. 50. How large is the angle which equals four times its complement?

Ex. 51. If $OC \perp OA$ and $OD \perp OB$, and if $\angle 2 = 70°$, compare $\angle 1$ and $\angle 3$.



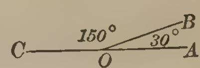
38. *Complements of the same angle or of equal angles are equal.*



The complement of m° is $(90 - m)$° and the complement of n° is $(90 - n)$°. If, now, $m = n$, then $90 - m$ must equal $90 - n$, for when equals are subtracted from equals the remainders are equal.

39. Two angles are **supplementary** if their sum is equal to a straight angle. Each of the angles is called the **supplement** of the other.

Thus, an angle of 150° is the supplement of an angle of 30°.

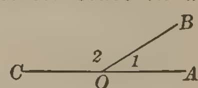


Ex. 52. What is the supplement of 80°? of 60°? of 100°? of 90°? of 0°? of x°? of $3x$°?

Ex. 53. How large is the angle which equals its supplement?

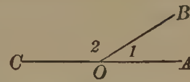
Ex. 54. How large is the angle which is nine times as large as its supplement?

40. *If two adjacent angles have their exterior sides in a straight line, they are supplementary.*



In the adjoining figure, $\angle 1$ and $\angle 2$ are adjacent angles. OC and OA are their exterior sides. If OC and OA lie in a straight line, then $\angle AOC = 1$ st. $\angle$. But $\angle 1 + \angle 2 = \angle AOC$. Hence, $\angle 1$ and $\angle 2$ are supplementary (§ 39).

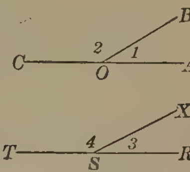
41. *If two adjacent angles are supplementary, their exterior sides lie in a straight line.*



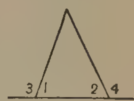
For, if $\angle 1 + \angle 2 = 1$ st. $\angle$, then $\angle AOC$ must be a straight angle and AC must be a straight line.

Ex. 55. Draw two adjacent angles which are not supplementary.

42. *Supplements of the same angle or of equal angles are equal.*



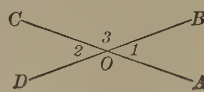
In the adjoining figure, let $\angle 1 = \angle 3$, and let AC and RT be straight lines. Then $\angle 2$ is the supplement of $\angle 1$ and $\angle 4$ is the supplement of $\angle 3$; that is, $\angle 2 = 180 - \angle 1$ and $\angle 4 = 180 - \angle 3$. Clearly, then, $\angle 2$ must equal $\angle 4$, for, when equal angles are subtracted from equal angles, the remainders are equal.



Ex. 56. In the adjoining figure, if $\angle 1$ equals $\angle 2$, then $\angle 3$ must equal $\angle 4$. Prove it.

Suggestion. What relation has $\angle 3$ to $\angle 1$? Why? What relation has $\angle 4$ to $\angle 2$? Why? Now use § 42.

43. Two angles are called **vertical angles** when the sides of one are prolongations through the vertex of the sides of the other. They are formed by two intersecting straight lines.



Thus, $\angle 1$ and $\angle 2$ are vertical angles.

Ex. 57. If $\angle 1 = 40°$, how many degrees are there in $\angle 3$?
How many degrees are there in $\angle 2 + \angle 3$?
How many degrees then are there in $\angle 2$?
How then do $\angle 2$ and $\angle 1$ compare?

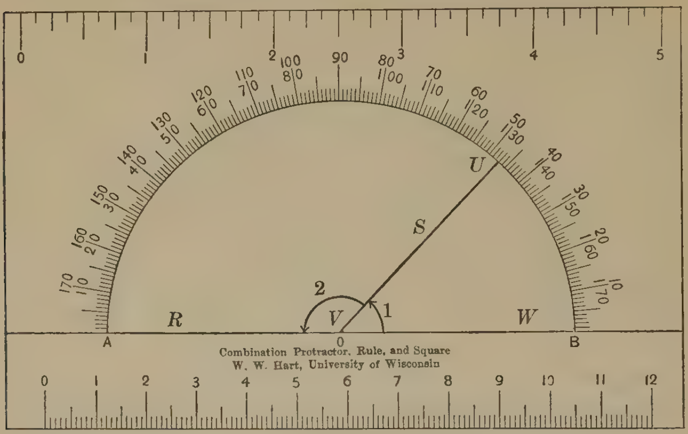
Ex. 58. (*a*) In the figure for § 43, suppose $\angle 1$ is 70°. How large is $\angle 3$? How large then is $\angle 2$?

(*b*) In the figure for § 43, suppose that $\angle 1$ is a right angle. How large is each of the other angles?

(*c*) How large is each of the other angles in § 43, if $\angle O$ contains $n°$?

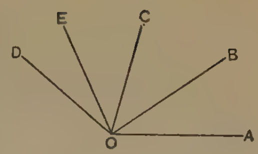
Ex. 59. Draw two straight lines that intersect. On transparent paper, make a tracing of one of the angles formed and compare it with its vertical angle. What do you conclude must be true about the vertical angles?

44. Experimental geometry. Many facts about geometrical figures can be discovered by measurement and observation of carefully drawn figures. It is necessary to use the **protractor**, a tool for measuring angles.



45. Using the protractor to measure an angle. Place the protractor so that its center O is on the vertex V of the angle, with the diameter AOB on the side VW, and the other side VS of the angle inside the semicircular cutout of the tool. VS then cuts the protractor scale, and the size of the angle can be read. In the figure, $\angle 1$ contains $47°$, and $\angle 2$ contains $133°$. Sometimes it is necessary to extend the side like VS, so that it will cut the protractor scale.

46. Using the protractor to draw a line making an angle of given size with a given line, at a point of that line. To draw a line making an angle of $47°$ with line RW at V, and lying above RW: Place the protractor with AB along RW, with O on V, and the semicircular cutout lying above RW. Opposite $47°$ on the outer scale, place a small point on the paper. Mark it U. Draw VU. Then $\angle WVU$ contains $47°$.



Ex. 60. Make on paper a tracing of the adjoining figure. On your paper, extend rays OA, OB, OC, OD, and OE until they are about 3 in. in length; then measure:

(*a*) $\angle AOB$; (*b*) $\angle AOC$; (*c*) $\angle BOE$; (*d*) $\angle COD$; (*e*) $\angle AOE$; (*f*) $\angle BOD$.

Ex. 61. (*a*) Draw a segment AB. From A, draw a ray above AB, making with AB an angle of 48°.

(*b*) From A, draw a ray below AB, making with AB an angle of 75°.

(*c*) From B, draw a ray above AB, making with AB an angle of 64°.

(*d*) From B, draw a ray below AB, making with AB an angle of 25°.

Ex. 62. Draw two rays from a point O, forming an angle of 50°. On each ray, locate a point which is 3 inches from O. Letter these points A and B. Measure $\angle ABO$ and also $\angle BAO$.

Ex. 63. On a line AB, at a point P, draw a ray PC, making $\angle APC = 80°$. Measure $\angle CPB$. What fact studied previously does this exercise verify?

Ex. 64. Draw two intersecting straight lines.

(*a*) Measure each of a pair of vertical angles. How do they compare?

(*b*) Measure each of the other pair of vertical angles. How do they compare?

Ex. 65. Let AB be any line segment. Draw $CA \perp AB$ at A, and $DB \perp AB$ at B. (See § 34.) Make $CA = DB$. Draw AD and CB. Compare them by measurement.

Ex. 66. Draw a segment AB 3 inches long. At its center, C, draw a line CD perpendicular to AB. From D draw DA and DB. Measure $\angle ADC$ and $\angle CDB$. What do you discover?

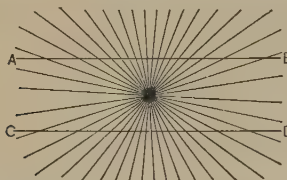
Ex. 67. Place on paper points A, B, and C, not all in one straight line (and not very close together). Draw segments AB, BC, and AC, forming *triangle* ABC. Measure $\angle A$, $\angle B$, and $\angle C$. What is their sum?

Ex. 68. (*a*) Draw segment AB, 2.5 inches long. At A, above AB, draw $AD \perp AB$, making it 2 inches long. At B, above AB, draw $BC \perp AB$, making it 3 inches long. Draw CD.

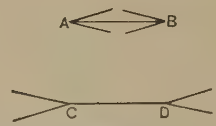
(*b*) Measure $\angle C$ and $\angle D$. What is their sum? What kind of angles are they?

(*c*) What is the sum of $\angle A$, $\angle B$, $\angle C$, and $\angle D$?

47. Objections to experimental geometry. To get satisfactory results, the figures must be drawn and measured with greater accuracy than is usually possible. Conclusions reached from the study of one or two special figures may be incorrect. *Frequently one is misled by assuming relations which appear to the eye to be correct.*



Are AB and CD straight lines?



Is AB equal to CD?

48. Demonstrative geometry. For the reasons given in § 47 and for other reasons, it is customary to study geometry by what is known as the demonstrative method. *Statements are not accepted until they are proved to be true,* except for a few which are assumed as a foundation.

49. An **axiom** is a general statement which is assumed to be true.

Ax. 1. *Quantities which are equal to the same quantity or to equal quantities are equal to each other.*

Ax. 2. *Any quantity may be substituted for its equal in a mathematical expression.*

Thus, if $x + y = z$ and $y = m$, then $x + m = z$.

Ax. 3. *If equals be added to equals, the sums are equal.*

Thus, if $x = y$ and $a = b$, then $x + a = y + b$.

Ax. 4. *If equals be subtracted from equals, the remainders are equal.*

Ax. 5. *If equals be multiplied by equals, the products are equal.*

Ax. 6. *If equals be divided by equals, the quotients are equal.*

Ax. 7. *The whole equals the sum of its parts.*

Ax. 8. *The whole is greater than any of its parts.*

50. A **postulate** is a geometrical statement which is assumed to be true.

Post. 1. *One and only one straight line can be drawn through two points.* (§ 5, *a*.)

Post. 2. *A straight line-segment can be extended infinitely far in two directions.* (§ 10.)

Post. 3. *The straight line-segment is the shortest line that can be drawn between two points.* (§ 13.)

Post. 4. *A straight line-segment has one and only one mid-point.* (§ 14.)

Post. 5. *A circle can be drawn with any point as center and any given segment as radius.* (§ 17.)

Post. 6. *An angle has one and only one bisector.* (§ 23.)

Post. 7. *All right angles are equal.* (§ 27.)

51. A **theorem** is a statement which requires proof.

52. Some **theorems** have been **proved informally.**

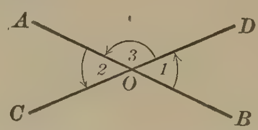
1. *Two straight lines can intersect at only one point.* (§ 9.)
2. *All radii of the same circle or of equal circles are equal.* (§ 19.)
3. *A straight angle equals two right angles.* (§ 30.)
4. *All straight angles are equal.* (§ 31.)
5. *The sum of all the successive adjacent angles around a point on one side of a straight line through the point is one straight angle.* (§ 35.)
6. *The sum of all the successive adjacent angles around a point in a plane is two straight angles.* (§ 36.)
7. *Complements of the same angle or of equal angles are equal.* (§ 38.)
8. *If two adjacent angles have their exterior sides in a straight line, they are supplementary.* (§ 40.)
9. *If two adjacent angles are supplementary, their exterior sides lie in a straight line.* (§ 41.)
10. *Supplements of the same angle or of equal angles are equal.* (§ 42.)

53. Formal demonstration of theorems. Every theorem can be expressed by a sentence which has one clause beginning with "if" and a second clause beginning with "then."

The clause beginning with "if" is called the **hypothesis**; it contains what is assumed or known. The other clause is called the **conclusion**; it states what is to be proved.

54. A formal demonstration may be arranged as below.

Theorem. *If two straight lines intersect, then vertical angles formed are equal.*



Hypothesis. Straight lines AB and CD intersect at O, forming vertical angles 1 and 2.

Conclusion. $\angle 1 = \angle 2$.

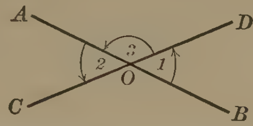
Proof:

STATEMENTS	REASONS
1. AB is a straight line.	1. By the hypothesis.
2. $\therefore \angle 1$ is a supp. of $\angle 3$.	2. If two adj. $\angle$s have their ext. sides in a st. line, they are supplementary.
3. CD is a straight line.	3. By the hypothesis.
4. $\therefore \angle 2$ is a supp. of $\angle 3$.	4. Same reason as No. 2.
5. $\therefore \angle 1 = \angle 2$.	5. Supplements of the same angle are equal.

Notice that each statement has beside it a proof of its correctness. *The hypothesis, a definition, an axiom or a postulate, or a previously proved theorem may be given as a "reason" or authority.* The authorities should be written *in full,* using abbreviations which have been agreed upon. The reason should start on the same line as the statement being proved, and should be numbered by the same number.

55. Another arrangement of a formal demonstration. *The teacher should insist upon the arrangement preferred.*

Theorem. *If two straight lines intersect, then vertical angles formed are equal.*



Hypothesis. Straight lines AB and CD intersect at O, forming vertical angles 1 and 2.

Conclusion. $\angle 1 = \angle 2$.

Proof. 1. AB is a straight line.

[By the hypothesis.]

2. $\therefore$ $\angle 1$ is a supplement of $\angle 3$.

[If two adjacent angles have their exterior sides in a straight line, they are supplementary.]

3. CD is a straight line.

[By the hypothesis.]

4. $\therefore$ $\angle 2$ is a supplement of $\angle 3$.

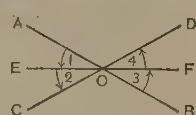
[Same reason as in Step 2.]

5. $\therefore$ $\angle 1 = \angle 2$.

[Supplements of the same angle are equal.]

Ex. 69. Prove in the same manner that $\angle AOD = \angle COB$.

Ex. 70. If a line bisects one of two vertical angles, it bisects the other also.



Hyp. $\measuredangle$s AOC and BOD are vertical angles. Line EOF bisects $\angle AOC$.

Con. Line EOF bisects $\angle BOD$.

Suggestion. Try to prove $\angle 3 = \angle 4$.

56. Besides the proofs of certain theorems, the methods of constructing certain figures are studied in geometry.

A **problem** is a construction to be made.

57. The word **proposition** is used to designate a theorem or a problem discussed in the text.

Review Questions

Ex. 71. How long is a line?

Ex. 72. How many lines can be drawn through one point?

Ex. 73. What is a straight line-segment?

Ex. 74. What are equal segments?

Ex. 75. What is the mid-point of a segment? How many mid-points can a segment have?

Ex. 76. What is a ray or half line? How long is it?

Ex. 77. What is an angle? Its vertex? Its sides?

Ex. 78. What are equal angles?

Ex. 79. What is the bisector of an angle?

Ex. 80. What is an acute angle? An obtuse angle? A straight angle? How many bisectors can an angle have?

Ex. 81. What are complementary angles? Illustrate.

Ex. 82. What are supplementary angles? Illustrate.

Ex. 83. What are adjacent angles? Illustrate.

Ex. 84. What are vertical angles? Illustrate.

Ex. 85. What are perpendicular lines? Illustrate.

Ex. 86. What is an axiom? Illustrate.

Ex. 87. What is a postulate? Illustrate.

Ex. 88. What is a theorem? What is its hypothesis? Its conclusion?

Ex. 89. How many straight lines can be drawn through two points?

Ex. 90. In how many points can two straight lines intersect?

Ex. 91. If two angles are equal, what is true of their complements?

Ex. 92. If two angles are equal, what is true of their supplements?

Ex. 93. Complete each of the following sentences:

(*a*) If two straight lines intersect, ⋯

(*b*) If two adjacent angles have their exterior sides in a straight line, ⋯

(*c*) If two adjacent angles are supplementary, then ⋯

(*d*) When one straight line meets another straight line, if two adjacent angles are equal, then ⋯

(*e*) All right angles are ⋯

(*f*) Quantities equal to the same or to equal quantities ⋯

(*g*) Any quantity may be substituted for ⋯

(*h*) The sum of all the successive adjacent angles around a point on one side of a straight line through the point is ⋯

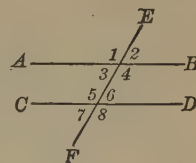
Optional Exercises

Ex. 94. What is the complement of 63°? Of 35° 40′? Of x°?

Ex. 95. What is the supplement of 88°? Of 70° 35′? Of y°?

Ex. 96. Two straight lines intersect so that one of the angles formed is 60°. How large is each of the other angles?

Ex. 97. In the adjoining figure, if $\angle 3 = \angle 7$, prove $\angle 2 = \angle 7$. Also prove $\angle 6 = \angle 3$.



Ex. 98. If $\angle 4 = \angle 5$, prove $\angle 1 = \angle 8$.

Ex. 99. If $\angle 3 = \angle 6$, prove $\angle 1 = \angle 8$.

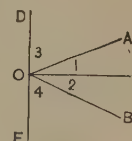
Suggestions. 1. What is the relation between $\angle 1$ and $\angle 3$?
2. What is the relation between $\angle 8$ and $\angle 6$?
3. Now use § 42.

Ex. 100. If $\angle 3 = \angle 7$, prove $\angle 1 = \angle 5$.

Ex. 101. If $\angle 1 = \angle 8$, prove $\angle 2 = \angle 7$.

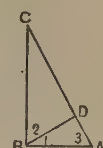
Ex. 102. If $\angle 1$ is the supplement of $\angle 6$, prove $\angle 2 = \angle 6$.

Ex. 103. Hyp. CO bisects $\angle BOA$. $DE \perp CO$.
Con. $\angle 3 = \angle 4$.



Suggestions. 1. What do you know about $\angle 1$ and $\angle 3$? About $\angle 2$ and $\angle 4$? 2. Use § 38.

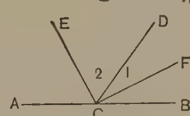
Ex. 104. Hyp. $\angle ABC$ is a rt. $\angle$.
$\angle 3$ is complementary to $\angle 1$.
Con. $\angle 3 = \angle 2$.



Ex. 105. Prove that the bisectors of two supplementary adjacent angles are perpendicular.

Hyp. $\angle BCD$ and $\angle DCA$ are supp.-adj. $\measuredangle$s.
CF bisects $\angle BCD$; CE bisects $\angle DCA$.

Con. $CF \perp CE$.



Suggestions. 1. $\angle 1 = \frac{1}{2} \angle BCD$. 2. $\angle 2 = ?$
3. $\therefore \angle 1 + \angle 2 = ?$

Ex. 106. If the bisectors of two adjacent angles are perpendicular, the angles must be supplementary. (See Fig. of Ex. 105.)

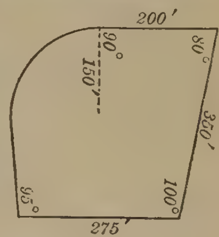
Suggestions. 1. $\angle 1 + \angle 2 = ?$ 2. $\angle BCD = 2 \times \angle 1$; $\angle DCA = ?$
3. $\angle BCD + \angle DCA = ?$

Ex. 107. If the bisectors of two adjacent angles make an angle of 45°, prove that the angles are complementary.

Practical Applications

Ex. 108. In order to walk across a field in a straight line, a man selects two objects in the direction in which he wishes to go, one of them directly between him and the other. As he walks, he keeps the first object between himself and the second. Why is this a good way to guide himself?

Ex. 109. An orchardist, setting out trees, first places a tree at each end of a row. How then may he locate the other trees of that row so that they will be in a straight line, without stretching a rope between the end trees?



Ex. 110. A man wishes a scale drawing of his suburban lot so that he may consult a landscape architect about the proper planting of it. He made the adjoining rough drawing of the lot, and then obtained the measurements indicated. Make a scale drawing of the lot, letting $\frac{1}{4}$ inch represent 25 feet.

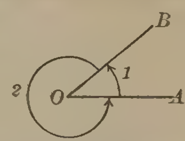
Supplementary Notes on Definitions

Note 1. *Point and straight line are undefined.* (See § 3 and § 4.) A definition describes a term by means of simpler terms. It is evident then that there must be some terms which cannot be defined, as there are no simpler terms by which to define them.

No definition of point can be given.

No satisfactory definition of straight line suitable for high school pupils can be given.

Hence point and straight line are left undefined.



Note 2. *Relating to the definition of an angle* (§ 21).

Two rays OA and OB *actually* form *two* angles; namely, $\angle 1$ and $\angle 2$ of the adjoining figure. In $\angle 1$, OA is the initial line (§ 21) and OB is the terminal line; in $\angle 2$, OB is the initial line and OA is the terminal line. Usually, one angle is less than and the other is greater than a straight angle.

Unless something is said to the contrary, $\angle AOB$ refers to the smaller angle formed by the rays OA and OB.

BOOK I

RECTILINEAR FIGURES

58. A **rectilinear figure** is composed only of straight lines.

59. Congruence. If asked to compare two sheets of paper as to shape and size, it is natural to place one upon the other to determine whether they can be made to coincide (fit together).

Geometrical figures are congruent (con-gru-ent) ($\cong$), **if they can be made to coincide.**

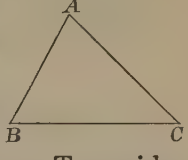
Historical Note. The symbol "$\cong$" was introduced by a mathematician, Leibnitz, about 1679.

60. Superposition is the process of placing one figure upon another for the purpose of comparing them. Literally, "superpose" means *place upon.*

Postulate. *A geometrical figure may be moved without changing its shape or the size of its parts.*

Ex. 1. Notice the panes of glass in your room. Are they probably congruent? Do they coincide now?

61. A **triangle** ($\triangle$) is the figure consisting of three points and the segments joining them by pairs; as $\triangle ABC$.



The points A, B, and C are the **vertices** of the triangle; AB, AC, and BC are the sides; $\angle A$, $\angle B$, and $\angle C$ are the angles. Two sides always *include* (form) an angle; and *we say* that two angles include a side.

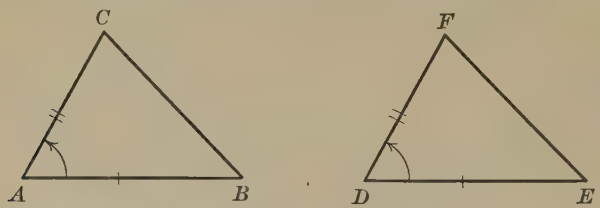
Ex. 2. Draw $\triangle ABC$, having $AB = 4$ in., $BC = 6$ in., and $\angle B = 50°$.

(*a*) Cut your triangle from the paper. Compare it by superposition with the triangles made by other members of your class.

(*b*) What do you conclude must be true about all triangles made according to the directions?

* PROPOSITION I. THEOREM

62. *If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are congruent.*



Hypothesis. In $\triangle ABC$ and $\triangle DEF$:

$$AB = DE; \quad AC = DF; \quad \angle A = \angle D.$$

Conclusion. $\triangle ABC \cong \triangle DEF$.

Plan. Try to prove them congruent by superposition.

Proof:

STATEMENTS	REASONS
1. Place $\triangle ABC$ on $\triangle DEF$ so point A is on point D, and AB is on DE.	1. A geometrical figure may be moved without changing its shape or the size of its parts.
2. Point B will fall on point E.	2. $AB = DE$, by hypothesis.
3. AC will fall on DF.	3. $\angle A = \angle D$, by hypothesis.
4. Point C will fall on point F.	4. $AC = DF$, by hypothesis.
5. BC will coincide with EF.	5. Only one straight line can be drawn through two points.
6. $\therefore \triangle ABC \cong \triangle DEF$.	6. Geometrical figures are congruent if they can be made to coincide.

Ex. 3. After you place $\triangle ABC$ so that A falls on D, does AB fall on DE, or must you place AB on DE?

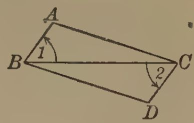
Ex. 4. Where would AC fall if $\angle A$ were less than $\angle D$?

Ex. 5. Where would C fall if AC were equal to $\frac{1}{2}$ DF?

Ex. 6. Make a drawing of the result if a $\triangle ABC$ is superposed on a $\triangle DEF$, when $AC = \frac{1}{2} DF$, $\angle A = \angle D$, and $AB = \frac{2}{3} DE$.

Note. This theorem may be abbreviated thus: s. a. s. = s. a. s.

63. Application of First Theorem. Illustrative Exercise.



Hypothesis. $AB = CD.$

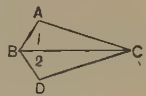
$\angle 1 = \angle 2.$

Conclusion. $\triangle ABC \cong \triangle BCD.$

Plan. Try to prove them congruent by s. a. s. = s. a. s.

Proof:

	STATEMENTS	REASONS
	In $\triangle ABC$ and $\triangle CBD$:	
1.	$AB = CD;$	1. By hypothesis.
2.	$\angle 1 = \angle 2;$	2. By hypothesis.
3.	$BC = BC.$	3. Any quantity equals itself.
4.	$\therefore \triangle ABC \cong \triangle CBD.$	4. If two $\triangle$ have two sides and the included $\angle$ of one equal respectively to two sides and the included $\angle$ of the other, the $\triangle$ are congruent.



Ex. 7. Hyp. $\angle 1 = \angle 2;\ AB = BD.$

Con. $\triangle ABC \cong \triangle BCD.$

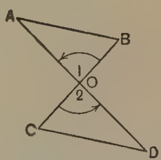


Fig. for Ex. 8, 9

Ex. 8. Hyp. AD and BC are st. lines.

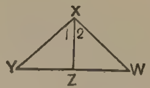
$AO = OD;\ BO = OC.$

Con. $\triangle ABO \cong \triangle CDO.$

Ex. 9. Hyp. O bisects $AD;\ AB = CD;$

$\angle A = 30°;\ \angle D = 30°.$

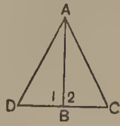
Con. $\triangle ABO \cong \triangle CDO.$



Ex. 10. Hyp. $XY = XW.$

XZ bisects $\angle X.$

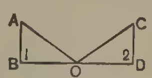
Con. $\triangle XYZ \cong \triangle XZW.$



Ex. 11. Hyp. DBC is a st. line.

$AB \perp DC;\ DB = BC.$

Con. $\triangle ADB \cong \triangle ABC.$



Ex. 12. Hyp. $AB \perp BD;\ CD \perp BD;$

O bisects $BD;\ AB = CD.$

Con. $\triangle\ ABO \cong \triangle ODC.$

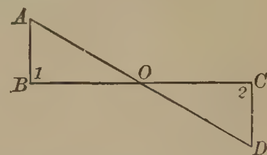
64. Corresponding sides or angles of congruent triangles are equal, since they coincide when the triangles have been made to coincide.

In congruent triangles, corresponding sides lie opposite equal angles and corresponding angles lie opposite equal sides.

Thus, in § 62, $\angle C$ corresponds to $\angle F$, and side BC corresponds to side EF. It follows that $\angle C = \angle F$ and $BC = EF$.

65. Fundamental Plan I. To prove that two segments are equal or two angles are equal, try to prove them corresponding parts of congruent triangles.

Illustrative Exercise



Hyp. BC is a st. line.
$AB \perp BC$; $DC \perp BC$;
$AB = DC$.
O is mid-point of BC.

Con. $AO = OD$.

Plan. Try to prove AO and OD corres. sides of cong. $\triangle$.

Proof:

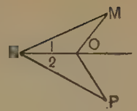
Statements	Reasons
1. $AB \perp BC$.	1. By hypothesis.
2. $\therefore \angle 1 =$ a rt. $\angle$.	2. Definition of $\perp$.
3. $DC \perp BC$.	3. By hypothesis.
4. $\therefore \angle 2 =$ a rt. $\angle$.	4. Definition of $\perp$.
5. $\therefore \angle 1 = \angle 2$.	5. All rt. $\angle$s are equal.
In $\triangle ABO$ and $\triangle DCO$:	
6. $AB = DC$.	6. By hypothesis.
7. $\angle 1 = \angle 2$.	7. By step 5.
8. $BO = OC$.	8. O is mid-pt. of BC, by hyp.
9. $\therefore \triangle ABO \cong \triangle DCO$.	9. If two $\triangle$ have two sides and the included $\angle$ of one equal respectively to two sides and the included $\angle$ of the other, the $\triangle$ are cong.
10. $\therefore AO = OD$.	10. Corres. sides of cong. $\triangle$ are equal.

Note. AO lies opposite $\angle 1$ and OD lies opposite $\angle 2$; hence they are corresponding sides of the congruent triangles.

Ex. 13. **Hyp.** NO bisects $\angle PNM$.

$MN = NP$.

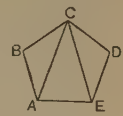
Con. $\angle M = \angle P$.



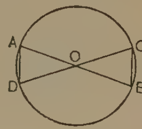
Ex. 14. **Hyp.** $AB = BC = CD = DE$.

$\angle B = \angle D$.

Con. $AC = CE$.



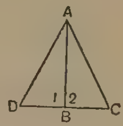
Ex. 15. If AB and CD are two diameters of a circle, prove that AD must equal BC.



Ex. 16. **Hyp.** DBC is a st. line.

$AD = AC$; $\angle DAB = \angle BAC$.

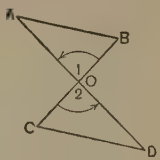
Con. $\angle D = \angle C$.



Ex. 17. **Hyp.** O bisects AD and BC.

Con. (a) $AB = CD$.

(b) $\angle A = \angle D$; $\angle B = \angle C$.



Ex. 18. **Hyp.** In the figure for Ex. 17:

O bisects AD; $AB = CD$;

$\angle A = \angle D$.

Con. (a) $OB = OC$; (b) $\angle B = \angle C$.

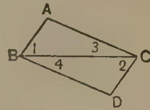
Ex. 19. **Hyp.** $AB = CD$.

$\angle 1 = \angle 2$.

Con. (a) $AC = BD$.

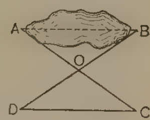
(b) $\angle A = \angle D$.

(c) $\angle 3 = \angle 4$.



Ex. 20. To obtain the distance AB.

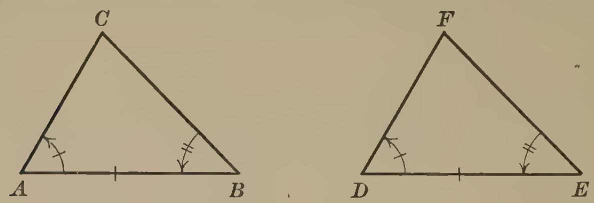
(1) Locate point O from which OA and OB may be measured. (2) Extend AO and BO, making $OC = AO$ and $OD = BO$. (3) Then $DC = AB$. Prove it.



Ex. 21. Draw a $\triangle ABC$, having $AB = 4$ in., $\angle A = 60°$, and $\angle B = 80°$. Cut your triangle from the paper and compare it with the triangles made by other members of your class. What do you conclude must be true about all triangles made according to the directions given?

* PROPOSITION II. THEOREM

66. *If two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, the triangles are congruent.*



Hypothesis. In $\triangle ABC$ and $\triangle DEF$:
$AB = DE$; $\angle A = \angle D$; $\angle B = \angle E$.

Conclusion. $\triangle ABC \cong \triangle DEF$.

Plan. Try to prove the $\triangle$ congruent by superposition.

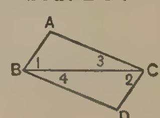
Proof:

STATEMENTS	REASONS
1. Place $\triangle ABC$ on $\triangle DEF$, with point A on point D, and side AB on side DE.	1. A geometrical figure may be moved without changing its shape or the size of its parts.
2. Point B will fall on point E.	2. $AB = DE$, by hypothesis.
3. AC will fall on DF, C falling somewhere on DF.	3. $\angle A = \angle D$, by hypothesis.
4. BC will fall on EF, C falling somewhere on EF.	4. $\angle B = \angle E$, by hypothesis.
5. $\therefore$ pt. C must fall on pt. F.	5. Two straight lines can intersect at only one point.
6. $\therefore$ $\triangle ABC \cong \triangle DEF$.	6. Geometrical figures are cong. if they can be made to coincide.

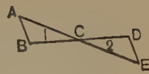
Ex. 22. Where would AC fall if $\angle A$ were less than $\angle D$? If greater?

Ex. 23. After proving $\triangle ABC$ congruent to $\triangle DEF$, what do you know about: (*a*) $\angle C$? Why? (*b*) About AC? (*c*) About BC?

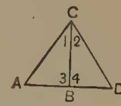
Ex. 24. Hyp. $\angle 1 = \angle 2$.
$\angle 3 = \angle 4$.
Con. $\triangle ABC \cong \triangle BCD$.



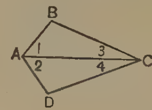
Ex. 25. Hyp. AE and BD are st. lines.
$\angle B = \angle D$; C bisects BD.
Con. $\angle A = \angle E$.



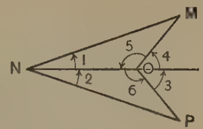
Ex. 26. Hyp. BC bisects $\angle C$.
$BC \perp AD$.
Con. $AB = BD$.



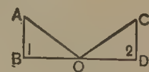
Ex. 27. Hyp. $\angle 1 = \angle 2$; $\angle 3 = \angle 4$.
Con. $\angle B = \angle D$.



Ex. 28. Hyp. NO bisects $\angle MNP$.
$\angle 4 = \angle 3$.
Con. $MN = NP$; $MO = OP$.
Suggestion. Prove $\angle 5 = \angle 6$, by § 42.



Ex. 29. Hyp. O bisects BD; $AB \perp BD$;
$CD \perp BD$.
$\angle AOB = 30°$; $\angle DOC = 30°$.
Con. $AO = OD$; $AB = CD$.



Note 1. Proposition II may be abbreviated thus: *a. s. a.* = *a. s. a.*
Note 2. Additional Exercises 1 to 6, p. 272, can be studied now.

67. A triangle is **scalene** when no two of its sides are equal; **isosceles** when two of its sides are equal; **equilateral** when all its sides are equal; **equiangular** when all its angles are equal.

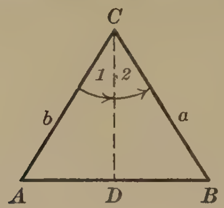
A triangle can be made to stand upon any one of its sides. Hence any side of a triangle can be considered its **base**; the opposite vertex is called **the vertex** of the triangle; and the angle at that vertex is called **the vertex angle** of the triangle.

In an isosceles triangle, the side which is not one of the two equal sides is called *the base*; then the angle formed by the equal sides is called *the vertex angle*, and the other two angles are called *the base angles*.

Ex. 30. (*a*) Draw an isosceles triangle ABC, having AB and BC each 3 inches long, and $\angle B$ 40°. Measure and compare the base angles.
(*b*) Repeat part (*a*), drawing any large isosceles triangle.

* PROPOSITION III. THEOREM

68. *The base angles of an isosceles triangle are equal.*



Hypothesis. In $\triangle ABC$, $AC = BC$.

Conclusion. $\angle A = \angle B$.

Plan. Try to prove $\angle A$ and $\angle B$ corres. $\angle$s of cong. $\triangle$s.

Proof:

STATEMENTS	REASONS
1. Let CD bisect $\angle C$, and meet AB at D.	1. An $\angle$ has a bisector.
In $\triangle ACD$ and $\triangle BCD$:	
2. $AC = BC$.	2. By hypothesis.
3. $CD = CD$.	3. Any quantity equals itself.
4. $\angle 1 = \angle 2$.	4. CD bisects $\angle C$; def.
5. $\therefore \triangle ACD \cong \triangle BCD$.	5. If two $\triangle$s have two sides and the included $\angle$ of one equal respectively to two sides and the included $\angle$ of the other, the $\triangle$s are congruent.
6. $\angle A = \angle B$.	6. Corres. sides of cong. $\triangle$s are equal.

Note. This theorem is ascribed to Thales.

69. A corollary is a theorem which is easily deduced from the theorem with which it is given. Draw a figure, form the hypothesis and conclusion, and give the proof, as usual.

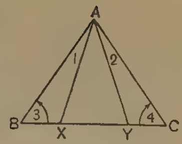
70. Cor. 1. *An equilateral triangle is also equiangular.*

71. Cor. 2. *The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to the base.*

Suggestion. In the figure above, to prove $CD \perp AB$, what must you prove about $\angle BDC$ and $\angle CDA$? How can you do that?

Ex. 31. **Hyp.** $AB = AC$.
$\angle 1 = \angle 2$.
Con. $\triangle ABX \cong \triangle AYC$.

Suggestion. Does $\angle 3 = \angle 4$? Why?

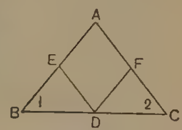


Ex. 32. In the figure for Exercise 31, let $AB = AC$, and $BY = CX$. Prove $AX = AY$.

Ex. 33. If $AB = AC$ and if D is the mid-point of BC, E of AB, and F of AC, prove that $ED = FD$.

Suggestions. Form the hypothesis and conclusion.

Read § 65. Does $BE = CF$? Why?



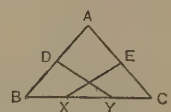
Ex. 34. After proving $DE = DF$ in Ex. 33, draw EF and prove that $\angle DEF = \angle DFE$.

Ex. 35. In the figure for Ex. 34, prove that $\angle AFE = \angle AEF$.

Ex. 36. In the figure for Exercise 33; if $AB = AC$; if D bisects BC; and if $\angle BDE = \angle CDF$; prove $\triangle DEF$ is isosceles.

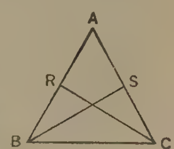
Suggestion. What must be proved about DE and DF? How can you do it?

Ex. 37. **Hyp.** $AB = AC$; D bisects AB;
E bisects AC; $BX = YC$.
Con. $XE = YD$.



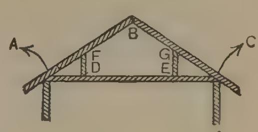
Ex. 38. **Hyp.** $AB = AC$; R bisects AB;
S bisects AC.
Con. $RC = BS$.

Suggestion. Try to prove $\triangle ABS \cong \triangle ARC$.



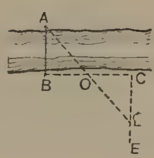
Ex. 39. Prove Exercise 38 again, using another pair of triangles.

Ex. 40. If AB and CB are two rafters of equal length in a roof, and if DF and EG are supports, perpendicular to the floor AC, at points equally distant from A and C respectively, prove that DF must equal EG. (From the hypothesis and conclusion first.)



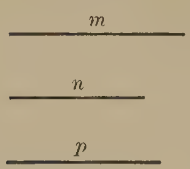
Ex. 41. To obtain the distance AB.

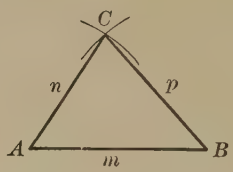
(1) Lay off $BC \perp$ to AB. (2) Lay off $CE \perp$ to BC. (3) Place a stake at O, the mid-point of BC. (4) Determine, by sighting, a point D on CE so that A, O, and D will be in the same straight line. Then $CD = AB$. Prove it.



PROPOSITION IV. PROBLEM

72. *Construct a triangle, having given its three sides.*





Given m, n, and p, the three sides of a triangle.

Required to construct the triangle.

Construction. 1. Draw $AB = m$.

2. With A as center, and n as radius, draw an arc.

3. With B as center, and p as radius, draw a second arc intersecting the first arc at C.

4. Draw AC and BC.

Statement. $\triangle ABC$ is the required triangle.

Proof. Its sides are of lengths m, n, and p, respectively.

Discussion. If one side is equal to or greater than the sum of the other two sides, the construction is impossible.

Ex. 42. Construct a triangle whose sides measure:
(*a*) 2 inches, 3 inches, and 3.5 inches respectively.
(*b*) 2.5 inches, 3 inches, and 2 inches respectively.
(*c*) 1 inch, 3 inches, and 2.5 inches, respectively.

Ex. 43. (*a*) Construct an isosceles triangle whose equal sides are 3 inches long, and whose base is 2 inches long.

(*b*) Measure its base angles and its vertex angle.

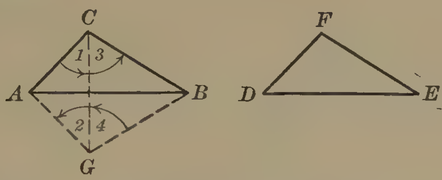
Ex. 44. Construct the equilateral triangle whose sides are 3 inches long. Measure each of its angles.

Ex. 45. Try to construct the triangle whose sides are 1 inch, 3 inches, and 2 inches respectively.

Ex. 46. Construct a triangle whose sides are 2 inches, 2.5 inches, and 3.5 inches respectively. Cut the triangle from the paper and compare it by superposition with the triangles made by other members of the class. What do you conclude must be true of all triangles made according to the directions given?

* PROPOSITION V. THEOREM

73. *If two triangles have the three sides of one equal respectively to the three sides of the other, the triangles are congruent.*



Hypothesis. In $\triangle ABC$ and $\triangle DEF$:
$AB = DE$; $BC = EF$; and $AC = DF$.

Conclusion. $\triangle ABC \cong \triangle DEF$.

Plan. Try to prove $\angle C = \angle F$, and use s. a. s. = s. a. s.

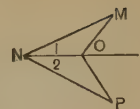
Proof:

STATEMENTS	REASONS
1. Place $\triangle DEF$ so that DE falls on AB, E falling on B, and so that F falls at G, on the opposite side of AB from C. Draw CG.	1. (*a*) A geometrical figure may be moved without changing its shape or the size of its parts. (*b*) Equal segments can be made to coincide.
2. In $\triangle ACG$, $AC = AG$.	2. $AG = DF = AC$, by hyp.
3. $\therefore \angle 1 = \angle 2$.	3. The base $\measuredangle$ of an isos. $\triangle$ are equal.
4. In $\triangle BCG$, $BC = BG$.	4. $BG = FE = BC$, by hyp.
5. $\therefore \angle 3 = \angle 4$.	5. Reason 3.
6. $\therefore \angle 1 + \angle 3 = \angle 2 + \angle 4$.	6. If equals be added to equals the sums are equal.
7. $\therefore \angle C = \angle G$, or $\angle C = \angle F$.	7. The whole equals the sum of its parts.
In $\triangle ABC$ and $\triangle DEF$:	
8. $AC = DF$; and $BC = EF$.	8. Why?
9. $\angle C = \angle F$.	9. See Step 7.
10. $\therefore \triangle ABC \cong \triangle DEF$.	10. § 62. See Note below.

Note 1. Give the authority without consulting § 62 if possible.

Note 2. This theorem may be abbreviated thus: s. s. s. = s. s. s.

Ex. 47. If $MN = NP$ and $MO = OP$, then
(*a*) $\triangle MNO \cong \triangle NPO$.
(*b*) NO bisects $\angle PNM$.



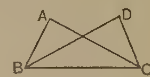
Ex. 48. If the opposite sides of a quadrilateral $ABCD$ are equal, then $\angle A = \angle C$.
(Form the Hyp., Con., and give the proof.)

Ex. 49. In quadrilateral $ABCD$ in Ex. 48, prove that $\angle B = \angle D$.

Ex. 50. On segment XY as base construct isosceles $\triangle XYZ$ and a second isosceles $\triangle XYW$. Draw ZW. Prove that $\triangle XZW \cong \triangle YZW$.

Ex. 51. In the adjoining figure, if $AB = DC$, and $AC = BD$, then $\angle A$ must equal $\angle D$.

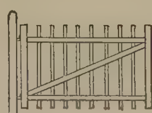
Suggestion. Prove $\triangle ABC \cong \triangle BCD$.

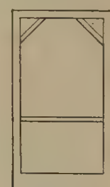


Ex. 52. Draw $\triangle ABC$, having $AB = AC$, and D bisecting BC.
Prove: (*a*) $\triangle ABD \cong \triangle ACD$; and (*b*) AD bisects $\angle BAC$.

Ex. 53. Practical applications of the fact that three sides determine a triangle are illustrated in the figures below. In each case, three lengths determine a triangle which makes some part of the object rigid. Why?


Roof Truss


Gate


Screen Door

Ex. 54. Why is a shelf bracket made in the form of a triangle?

Ex. 55. Can you give any other practical uses of Proposition V?

Ex. 56. State three theorems by which two triangles can be proved congruent.

Ex. 57. What are corresponding parts of congruent triangles?

Ex. 58. State Fundamental Plan I.

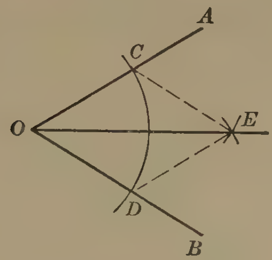
Ex. 59. If $\triangle XYZ$ is equilateral, and points A, B, and C are located on XY, YZ, and XZ respectively, so that $XA = YB = ZC$, prove $AB = BC = AC$.

Ex. 60. (*a*) What is *the* base of an isosceles triangle?
(*b*) Construct the isosceles triangle whose base is 1 inch long and whose equal sides are 2 inches long.

Note. Additional Exercises 7 to 9, p. 272, can be studied now.

PROPOSITION VI. PROBLEM

74. *Bisect a given angle.*



Given $\angle BOA$.

Required to bisect $\angle BOA$.

Construction. 1. With O as center and a convenient radius, draw an arc intersecting AO at C and BO at D.

2. With C and D as centers and with equal radii, draw arcs intersecting at E.

3. Draw OE.

Statement. OE bisects $\angle BOA$.

Proof:

STATEMENTS	REASONS
1. Draw CE and DE.	1. A line can be drawn through 2 pts.
In $\triangle OCE$ and $\triangle ODE$:	
2. $OC = OD$.	2. Radii of the same ⊙ are equal.
3. $CE = DE$.	3. Radii of equal ⓢ are equal.
4. $OE = OE$.	4. Why?
5. $\therefore \triangle OCE \cong \triangle ODE$.	5. Why?
6. $\therefore \angle BOE = \angle EOA$.	6. Why?
7. $\therefore OE$ bisects $\angle BOA$.	7. Definition of bisector.

Discussion. The construction is always possible.

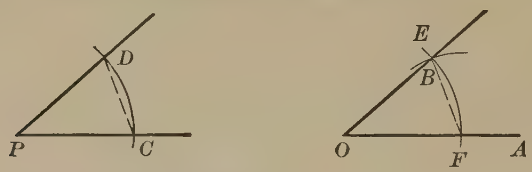
Ex. 61. Draw an obtuse angle. Divide it into four equal parts.

Ex. 62. Construct the bisectors of the three angles of a large triangle. What seems to happen?

Ex. 63. Draw two intersecting straight lines. Bisect each of the four angles formed.

PROPOSITION VII. PROBLEM

75. *At a given point in a line, construct a line making a given angle with a given line.*



Given $\angle P$, and point O in line OA.

Required to construct a line from O, making with OA an angle equal to $\angle P$.

Construction. 1. With P as center and a convenient radius, draw an arc intersecting the sides of $\angle P$ at C and D. Draw CD.

2. With O as center and PC as radius, draw arc FE.

3. With F as center and CD as radius, draw an arc intersecting arc FE at B.

4. Draw OB.

Statement. $\angle AOB = \angle P$.

Plan. Use § 65. Draw FB. Prove $\triangle FOB \cong \triangle CPD$.

(Proof left to the pupil.)

Ex. 64. Construct $\triangle ABC$ with side AB = 4 in., and $\angle A$ and $\angle B$ equal to the angles given in the adjoining figure. Should the triangles made by different pupils be congruent?

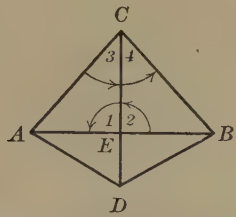
Ex. 65. Draw an acute angle. Construct an angle equal to twice this acute angle.

76. A line perpendicular to a segment at its mid-point is the **perpendicular-bisector** of the segment.

Ex. 66. Draw a line BC, 3 in. in length. With compasses, locate above BC a point A which is 2 in. from B and 2 in. from C. Locate similarly a point D below BC which is 3 in. from B and 3 in. from C. Draw AD, cutting BC at E. (*a*) Compare BE with EC. (*b*) Measure $\angle BEA$. (*c*) What kind of lines are AD and BC?

PROPOSITION VIII. THEOREM

77. *If two points are each equidistant from the ends of a segment, they determine the perpendicular-bisector of the segment.*



Hypothesis. C and D are equidistant from the ends of segment AB. CD intersects AB at E.

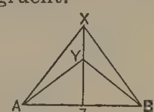
Conclusion. CD is the perpendicular-bisector of AB.

Plan. Try to prove $AE = EB$, and $\angle 1$ and $\angle 2$ are rt. $\angle$s.

Proof:

STATEMENTS	REASONS
In $\triangle ACD$ and $\triangle CDB$:	
1. $AC = CB$, and $AD = DB$.	1. By the hypothesis.
2. $CD = CD$	2. Why? (Give the reason.)
3. $\therefore \triangle ACD \cong \triangle CDB$.	3. Why?
4. $\therefore \angle 3 = \angle 4$.	4. Why?
5. $\triangle ACE \cong \triangle ECB$.	5. Give the full proof.
6. $\therefore AE = EB$.	6. Why?
7. Also $\angle 1 = \angle 2$.	7. Why?
8. $\therefore$ $\angle$s 1 and 2 are rt. $\angle$s.	8. § 26. Give reason in full.
9. $\therefore$ CD is $\perp$-bis. of AB.	9. Definition of $\perp$-bis. (§ 76).

Note. It is often necessary to prove one pair of triangles congruent in order to obtain two equal angles or two equal segments which are required in turn to prove another pair of triangles congruent.



Ex. 67. If XZ is the perpendicular-bisector of AB, and Y is a point on XZ, prove

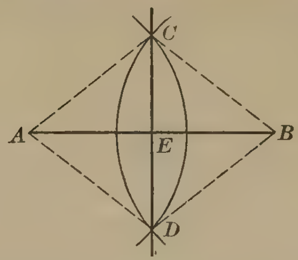
$$\triangle AXY \cong \triangle BXY.$$

Plan. Prove $\triangle AXZ \cong \triangle BXZ$ to get $AX = XB$; then prove $\triangle AZY \cong \triangle BZY$ to get $AY = YB$.

Note. Additional Exercises 10–13, p. 273, can be studied now.

PROPOSITION IX. PROBLEM

78. *Construct the perpendicular-bisector of a given segment.*



Given line segment AB.

Required to construct the perpendicular-bisector of AB.

Construction. 1. With A and B as centers, and with equal radii, draw arcs intersecting at C and also at D.
2. Draw CD intersecting AB at E.

Statement. CD is the perpendicular-bisector of AB.

Plan. Use § 77.

Proof:

STATEMENTS	REASONS
1. $AC = BC$.	1. Radii of equal ⓢ are equal.
2. $AD = BD$.	2. Why?
3. $\therefore CD$ is $\perp$-bis. of AB.	3. (Give reason in full.)

Discussion. The construction is always possible.

This same construction is used to bisect a segment.

Ex. 68. Draw a triangle of large size. Construct the perpendicular-bisectors of its three sides. What happens?

Ex. 69. Divide a given segment into four equal parts.

79. A **median** of a triangle is the line drawn from a vertex to the mid-point of the opposite side.

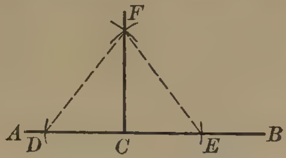
Ex. 70. Draw a triangle of large size. Construct the three medians of the $\triangle$. What happens?

Ex. 71. Prove that the median drawn to the base of an isosceles triangle bisects the vertical angle. (Construct the figure.)

Note. Additional Exercises 14–16, p. 273, can be studied now.

PROPOSITION X. PROBLEM

80. *At a point in a line, construct a perpendicular to the line.*



Given C, any point in line AB.

Required to construct a perpendicular to AB at C.

Construction. 1. With C as center, and any radius, draw arcs intersecting AB at D and E respectively.

2. With D and E as centers and a radius greater than one half DE, draw two arcs intersecting at F.

3. Draw CF.

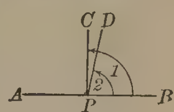
Statement. $CF \perp AB$ at C.

Proof:

	STATEMENTS		REASONS
1.	Draw DF and EF.	1.	A line can be drawn through 2 points.
2.	$DC = EC$.	2.	Why?
3.	$DF = EF$.	3.	Why?
4.	$\therefore CF$ is $\perp$-bis. of DE.	4.	§ 77. Give reason in full.
5.	$\therefore CF \perp AB$.	5.	Since AB and DE are the same straight line.

Discussion. The construction is always possible.

81. *Only one perpendicular can be drawn to a line at a point in the line.* For, if CP and DP were both perpendicular to AB at P, then $\angle 1$ and $\angle 2$ would both be right angles and hence would be equal. But $\angle 1$ is greater than $\angle 2$, for the whole is greater than any of its parts.

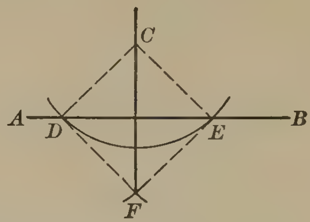


Ex. 72. At X, Y, and Z, of line AB, construct perpendiculars to AB.

Ex. 73. Prove CF perpendicular to DE in § 80 by proving that $\angle FCD = \angle FCE$, and then using § 26.

PROPOSITION XI. PROBLEM

82. *Construct a perpendicular to a line from a point not in the line.*



Given line AB and point C not in AB.

Required to construct a $\perp$ to AB from C.

Construction. 1. With C as center and a convenient radius, draw an arc intersecting AB at D and E respectively.

2. With D and E as centers, and equal radii, draw two arcs intersecting at F.

3. Draw CF.

Statement. $CF \perp AB$.

Proof:

STATEMENTS	REASONS
1. Draw CD, CE, DF, and EF.	1. A line can be drawn through 2 points.
2. $DC = EC$.	2. Why? (Give full reason.)
3. $DF = EF$.	3. Why?
4. $\therefore CF$ is $\perp$-bis. of DE.	4. Why?
5. $\therefore CF \perp AB$.	5. AB and DE are the same straight line.

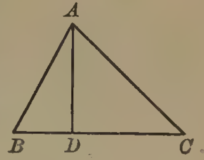
Discussion. The construction is always possible.

Historical Note. This construction is attributed to Oenipodes of Chios (465 B.C.)

Ex. 74. Draw a line AB. Place points X and Y above AB, and Z below AB. Construct perpendiculars to AB from X, Y, and Z.

Note. Additional Exercises 17–18, p. 273, can be studied now.

83. The **distance from a point to a line** is the length of the perpendicular from the point to the line.



84. The **altitude of a triangle** is the perpendicular drawn from a vertex to the opposite side, or the opposite side extended; as AD.

Ex. 75. How many altitudes does a triangle have?

Ex. 76. Draw a triangle of large size. Construct its three altitudes. What happens?

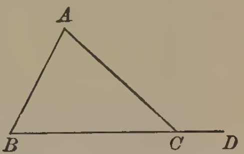
Ex. 77. Draw a line RS. From A and B, on opposite side of RS, construct perpendiculars to RS. Measure them.

Ex. 78. Construct $\triangle ABC$, having $AB = 2$ in., $BC = 2.5$ in., and $AC = 3.5$ in.

(*a*) Construct the altitude to AC from B; measure it.
(*b*) Construct the altitude to AB from C; measure it.

Ex. 79. Construct $\triangle XYZ$, having $XY = 3$ in., $XZ = 2$ in., and $YZ = 2.5$ in.

(*a*) Construct the perpendicular bisector of XY.
(*b*) Construct the median to XY. Measure it.
(*c*) Construct the altitude to XY. Measure it.



85. An **exterior angle** of a triangle is the angle at any vertex formed by a side of the triangle and the adjacent side extended; as, $\angle DCA$.

One interior angle is *adjacent* to the exterior angle and the other two are *remote interior* angles. Thus, $\angle A$ and $\angle B$ are the remote interior angles of exterior $\angle DCA$.

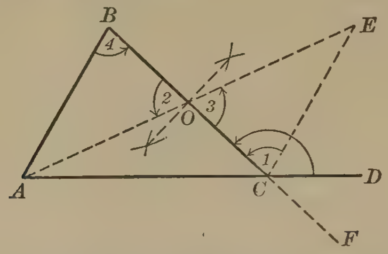
Ex. 80. Draw a large figure like that in § 85. Measure the exterior $\angle DCA$ and each of the remote interior angles. How does the exterior angle compare with the remote interior angles?

Ex. 81. In the figure for Prop. IX, p. 40, $\angle CEB$ is an exterior angle of what triangles?

Note. Straightedge and compasses alone are employed in making the constructions in elementary geometry. Naturally some constructions cannot be made with these tools alone. For example, *it is impossible to trisect an angle by ruler and compasses alone.*

PROPOSITION XII. THEOREM

86. *An exterior angle of a triangle is greater than either remote interior angle.*



Hypothesis. $\angle DCB$ is an exterior $\angle$ of $\triangle ABC$.

Conclusion. $\angle DCB > \angle B$; also $\angle DCB > \angle A$.

Plan. Try to prove $\angle DCB >$ an $\angle$ which equals $\angle B$.

Proof:

STATEMENTS	REASONS
1. Let O bisect BC. Draw AO. Extend AO to E, making $OE = AO$. Draw CE.	1. A segment has a mid-point. A segment can be extended.
2. $\triangle ABO \cong \triangle OCE$.	2. By s. a. s. = s. a. s. (See Note)
3. $\therefore \angle 4 = \angle 1$.	3. Give reason.
4. But $\angle DCB > \angle 1$.	4. Ax. 8, § 49, p. 18.
5. $\therefore \angle DCB > \angle 4$.	5. Ax. 2, § 49, p. 18.
6. Extend BC to F.	6. A segment can be extended.
7. $\angle ACF > \angle BAC$, or $\angle A$.	7. By a proof like Steps 1 to 5.
8. But $\angle DCB = \angle ACF$.	8. Why?
9. $\therefore \angle DCB > \angle A$.	9. Ax. 2, § 49.

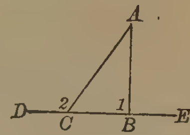
Note. Prepare to give the full proof and authorities. Give in the "statements" column, the sides and angle by which the triangles are proved congruent, and, in the "reasons" column, give the authorities.

Ex. **82.** In the figure above: (*a*) compare $\angle EOB$ with $\angle 4$; (*b*) compare $\angle AOC$ with $\angle 1$; (*c*) compare $\angle FCE$ with $\angle 3$.

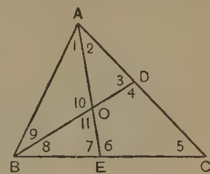
Ex. **83.** Give the details of the proof of Step 7, above.

Suggestion. Let X be the mid-point of AC. Draw BX. Extend BX to Y, making $XY = BX$. Draw YC. Prove $\triangle BAX \cong \triangle XYC$.

87. *Cor. *There can be only one perpendicular to a given line from a point not on the line.*



If $AB \perp DE$, then AC cannot be $\perp DE$, for $\angle 2 > \angle 1$ and hence $\angle 2$ is an obtuse angle.



Ex. 84. In the adjoining figure, what angles are less than $\angle 10$? Why?

Similarly prove $\angle 6$ greater than certain angles; also $\angle 3$, $\angle 11$, etc.

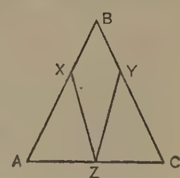
Review Exercises

Ex. 85. Construct $\triangle ABC$ having $AB = 1.75$ in., $\angle A = 90°$, and $AC = 2.5$ in.

Ex. 86. Construct $\triangle ABC$, having $AB = 2$ in., $\angle B = 45°$, and $BC = 1.5$ in.

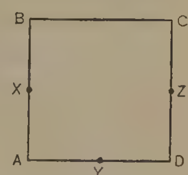
Ex. 87. Draw an acute and an obtuse angle. Construct an angle equal to their sum.

Ex. 88. When are two triangles congruent? State the three theorems by which you can now prove triangles congruent.



Ex. 89. In $\triangle ABC$, $AB = BC$; $BX = BY$; Z bisects AC. Prove $XZ = YZ$.

Ex. 90. In the figure for Ex. 89, draw XC and YA. Prove $XC = YA$.



Ex. 91. In a square, as you know, the sides are all equal and the angles are all right angles.

In the adjoining square, let X bisect AB, Y bisect AD, and Z bisect CD. Prove $XY = YZ$.

Ex. 92. Also, prove $XD = AZ$.

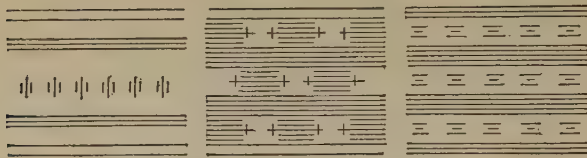
Ex. 93. Also, prove $BY = CY$.

Ex. 94. Also, prove $CX = BY$.

Ex. 95. (*a*) Construct $\triangle ABC$, having its sides each 3 inches long.
(*b*) Construct the perpendicular bisector of AB, locating the mid-point X of AB. Similarly locate the mid-point Y of BC.

(*c*) Construct the perpendicular to AC from X, meeting it at W. Similarly, construct the perpendicular to AC from Y, meeting it at R.

PARALLEL LINES

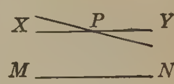


Some Parallel Line Border Designs

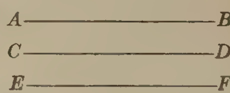
88. Two straight lines are **parallel** (||) if they lie in the same plane and do not meet even if extended.

Two different straight lines in the same plane therefore either intersect or are parallel lines.

In plane geometry, it is assumed that all the points and lines lie in a plane; therefore the phrase "in the same plane" is omitted from most statements.



89. Postulate of Parallels. *Through a given point there can be only one parallel to a given line.* Thus, through P only $XY \parallel MN$.



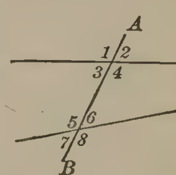
90. Cor. *If two lines are parallel to a third line, they are parallel to each other.*

Hyp. $AB \parallel CD$; $EF \parallel CD$.

Con. $AB \parallel EF$.

Proof. If AB and EF are not parallel, they must meet at a point. Through this point there would then be two lines parallel to CD. But this is impossible by the Postulate of Parallels. Hence AB must be parallel to EF.

91. If two lines are cut by a third line, AB, called a **transversal**, the angles are named as follows:



∠s 3, 4, 5, and 6 are called **interior angles.**

∠s 1, 2, 7, and 8 are called **exterior angles.**

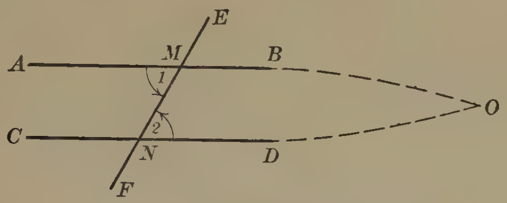
∠s 3 and 6 are called **alternate-interior angles**; also ∠s 5 and 4.

∠s 1 and 8 are called **alternate-exterior angles**; also ∠s 2 and 7.

∠s 2 and 6 are called **corresponding angles**; also ∠s 1 and 5, ∠s 3 and 7, and ∠s 4 and 8.

* PROPOSITION XIII. THEOREM

92. *If two lines are cut by a transversal so that a pair of alternate-interior angles are equal, the lines are parallel.*



Hypothesis. EF cuts AB at M, and CD at N; $\angle 1 = \angle 2$.

Conclusion. $AB \parallel CD$.

Plan. Try to prove AB and CD cannot meet.

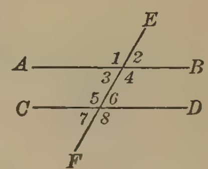
Proof:

STATEMENTS	REASONS
1. Suppose AB not $\parallel CD$. Suppose AB meets CD at O, on right of EF.	1. Two lines in the same plane either are parallel or intersect.
2. $\therefore \angle 1$ is an ext. $\angle$ of $\triangle MNO$.	2. Definition of ext. $\angle$.
3. $\therefore \angle 1 > \angle 2$.	3. § 86, p. 44
4. But $\angle 1 = \angle 2$.	4. By the hypothesis.
5. $\therefore AB$ cannot meet CD on the right of EF.	5. Since assuming AB meets CD on the right of EF leads to a contradiction. See the remarks below.
6. Similarly AB cannot meet CD on the left of EF.	6. By a proof like that in Steps 1 to 5.
7. $\therefore AB \parallel CD$.	7. Two straight lines are $\parallel$ if they lie in the same plane and do not meet even if extended.

Note. The argument in Step 5 is *that Steps 3 and 4 indicate that something is wrong.* Step 4 is correct. Steps 2 and 3 are correct if AB meets CD on the right of EF. Since there is an error, therefore the assumption that AB and CD meet on the right of EF must be wrong. Therefore AB does not meet CD on the right of EF.

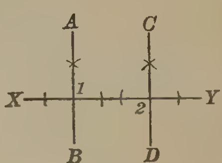
93. The method of proof used in §§ 90 and 92 is called the **Indirect Method of Proof.** Notice: (*a*) it starts by assuming the negative of the conclusion; (*b*) it follows up the consequences of this assumption until a statement is reached which contradicts a known fact; (*c*) this contradiction is made the basis for asserting that the desired conclusion is true.

94. Fundamental Plan II. To prove two lines parallel, try to prove a transversal of them makes a pair of alternate-interior angles equal.



95. Cor. 1. *If two lines are cut by a transversal so that a pair of corresponding angles are equal, the lines are parallel.*

Hyp. AB and CD are cut by EF; $\angle 2 = \angle 6$.
Con. $AB \parallel CD$.
Plan. Try to prove $\angle 3 = \angle 6$. Then use § 92.



96. Cor. 2. *If two lines are perpendicular to a third line, they are parallel.*

Hyp. $AB \perp XY$; $CD \perp XY$.
Con. $AB \parallel CD$.
Plan. Try to prove $\angle 1 = \angle 2$. Then use § 92.

97. Cor. 3. *If two lines are cut by a transversal so that a pair of interior angles on the same side of the transversal are supplementary, the lines are parallel.* (Use Fig. § 95.)

Hyp. AB and CD are cut by EF; $\angle 4 + \angle 6 = 1$ st. $\angle$.
Con. $AB \parallel CD$.
Plan. Try to prove $\angle 3 = \angle 6$.

Proof.		
1.	$\angle 3$ is supp. of $\angle 4$.	§ 40.
2.	$\angle 6$ is supp. of $\angle 4$.	Hyp.
3.	$\therefore \angle 3 = \angle 6$.	Why?
4.	$\therefore AB \parallel CD$.	Why?

Ex. 96. Prove Cor. 1, above, if $\angle 3 = \angle 7$; also if $\angle 1 = \angle 5$.

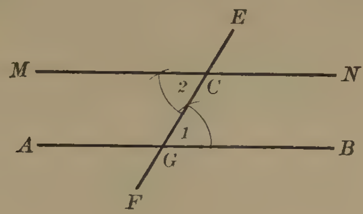
Ex. 97. Prove Cor. 3, above, if $\angle 3 + \angle 5 = 1$ st. $\angle$.

Ex. 98. Prove $AB \parallel CD$, in § 95, if $\angle 1 = \angle 8$.

Ex. 99. Prove $AB \parallel CD$ if $\angle 1$ and $\angle 7$ are supplementary.

PROPOSITION XIV. PROBLEM

98. *Construct a parallel to a line through a point not in the line.*



Given line AB and C, any point not in line AB.

Required to construct a parallel to AB through C.

Construction. 1. Draw FE through C, meeting AB at G.
2. At C, construct MCN making $\angle 2 = \angle 1$.

Statement. $MN \parallel AB$.

(Proof to be given by the pupil.)

Discussion. The construction is always possible.

Note. MN can be constructed $\parallel AB$ by making $\angle ECN = \angle 1$.

Ex. 100. Draw acute $\angle XYZ$. On YZ, place points A, B, and C so that $YA = 1$ in., $YB = 1.5$ in., $YC = 2$ in.

(*a*) At A, construct a line parallel to XY.

(*b*) Similarly construct parallels to XY at B and C.

Ex. 101. Draw an acute $\angle XAY$. On XA, place point B so that $AB = 1.5$ in.; on AY, place point C so that $AC = 1.5$ in.

(*a*) Through C, construct $CM \parallel AB$.

(*b*) Through B, construct $BK \parallel AC$, meeting CM at D.

(*c*) Measure CD and BD.

Ex. 102. In the figure for Prop. XII, p. 44, prove $CE \parallel AB$.

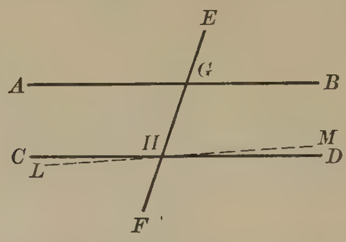
Ex. 103. Construct the figure for § 96; then construct the bisectors $\angle$s 1 and 2. Prove that these bisectors are parallel.

Ex. 104. If sides BA and CA of any $\triangle ABC$ are extended their own lengths through vertex A to D and E respectively, then DE is parallel to BC. *Suggestion.* Apply § 94 and § 65.

Note. Additional Exercises 19–23, p. 273, can be studied now.

* PROPOSITION XV. THEOREM

99. *If two parallels are cut by a transversal, alternate-interior angles are equal.*



Hypothesis. EF cuts $\|^s$ AB and CD at G and H.

Conclusion. $\angle AGH = \angle GHD$.

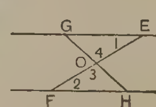
Plan. Use the indirect method of proof.

Proof: STATEMENTS	REASONS
1. Suppose $\angle AGH < \angle DHG$.	1. A possibility.
2. Draw LM through H so that $\angle MHG = \angle AGH$.	2. This is possible since $\angle AGH$ is less than $\angle DHG$.
3. $\therefore LHM \parallel AB$.	3. § 92, p. 47.
4. But this is impossible, since $CHD \parallel AB$, by hypothesis.	4. Through a given point, there can be only one parallel to a line.
5. $\therefore \angle AGH$ is not $< \angle DHG$.	5. Since Step 1 leads to a contradiction.
6. Also $\angle AGH$ is not $> \angle DHG$.	6. By a similar proof.
7. $\therefore \angle AGH = \angle DHG$.	7. Since $\angle AGH$ is not $<$ or $> \angle DHG$.

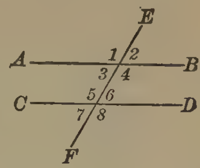
Note. Review §93, p. 48.

Ex. 105. If EF cuts parallels AB and CD so that $\angle AGH = 30°$, how many degrees are there in each of the other angles of the figure?

Ex. 106. If EF, joining two parallels, be bisected and GH be drawn through the mid-point and included between the parallels, then GH will also be bisected by the point.



100. Cor. 1. *If two parallels are cut by a transversal, corresponding angles are equal.*



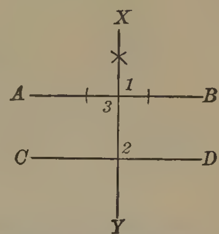
Hyp. $AB \parallel CD$; $\angle 2$ and $\angle 6$ are corresponding angles

Con. $\angle 2 = \angle 6$.

Suggestion. Compare $\angle 2$ and $\angle 6$ with $\angle 3$.

Ex. 107. If $AB \parallel CD$ in the figure of § 100, prove $\angle 3 = \angle 7$; also prove $\angle 1 = \angle 5$; also prove $\angle 4 = \angle 8$.

101. Cor. 2. *If a line is perpendicular to one of two parallels, it is perpendicular to the other also.*



Hyp. $AB \parallel CD$; $XY \perp AB$.

Con. $XY \perp CD$.

[What must be proved about $\angle 2$?]

102. Cor. 3. *If two parallels are cut by a transversal, interior angles on the same side of the transversal are supplementary.*

Hyp. $AB \parallel CD$; $\angle 4$ and $\angle 6$ are int. $\measuredangle$ on the same side of the transversal. (Fig. § 100.)

Con. $\angle 4 + \angle 6 = 1$ st. $\angle$.

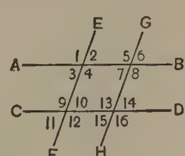
Proof. 1. $\angle 4 + \angle 3 = 1$ st. $\angle$.	1. § 40
2. $\angle 6 = \angle 3$.	2. Why?
3. $\therefore \angle 4 + \angle 6 = 1$ st. $\angle$.	3. Ax. 2, § 49

Ex. 108. If $AB \parallel CD$ in the figure of § 100, prove $\angle 3 + \angle 5 = 1$ st. $\angle$.

Ex. 109. If $AB \parallel CD$ in the figure of § 100, prove $\angle 1 + \angle 6 = 1$ st. $\angle$.

Ex. 110. If $AB \parallel CD$ in the figure of § 100, prove $\angle 4 + \angle 7 = 1$ st. $\angle$.

Ex. 111. In the adjoining figure, if $AB \parallel CD$, and $EF \parallel GH$, prove: (*a*) $\angle 1 = \angle 13$;
(*b*) $\angle 2 = \angle 15$; (*d*) $\angle 4 + \angle 14 = 1$ st. $\angle$.
(*c*) $\angle 7 = \angle 10$; (*e*) $\angle 5 + \angle 11 = 1$ st. $\angle$.



Ex. 112. If a line be drawn parallel to the base of an isosceles triangle, cutting the two sides of the triangle, it makes equal angles with these sides.

PROPOSITION XVI. THEOREM

103. *If two angles have their sides respectively parallel, they are equal, provided both pairs of parallels extend in the same directions from their vertices, or in opposite directions.*

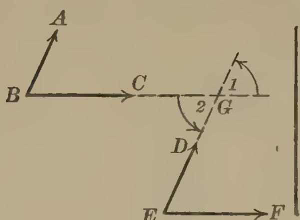


FIG. 1

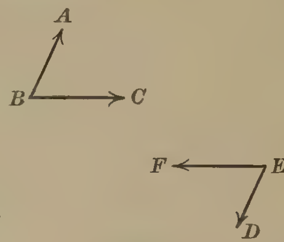


FIG. 2

I. (Fig. 1) **Hypothesis.** $\angle CBA$ and $\angle FED$ have $AB \parallel DE$ and $BC \parallel EF$, with the sides extending in the same directions * from the vertices.

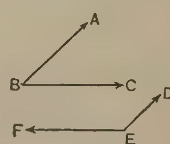
Conclusion. $\angle CBA = \angle FED$

Plan. Extend BC and ED until they intersect at G. Compare $\angle B$ and $\angle E$ with $\angle 2$.

II. (Fig. 2) **Hypothesis.** $\angle CBA$ and $\angle FED$ have $AB \parallel DE$ and $BC \parallel EF$, with the sides extending in opposite directions from the vertices.

Conclusion. $\angle CBA = \angle FED$.

Plan. Extend BC and DE until they intersect at a point O.

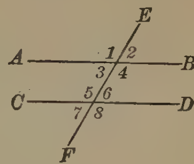


Ex. 113. If two angles have their sides respectively parallel, one pair of parallels extending in the same directions but the other pair extending in opposite directions from their vertices, the angles are supplementary. (Prove $\angle B + \angle E = 1$ st. $\angle$.)

* Note. The sides extend in the *same* direction if they are on the same side of a straight line joining their vertices, and in *opposite* directions if they are on *opposite* sides of this line.

104. One theorem is called the **converse** of another when an essential statement of the hypothesis and conclusion of the one become statements in the conclusion and hypothesis of the other. Thus, Prop. XV is the converse of Prop. XIII.

In Proposition XIII	In Proposition XV
Hyp. EF cuts AB and CD. $\angle 3 = \angle 6$.	**Hyp.** EF cuts AB and CD. $AB \parallel CD$.
Con. $AB \parallel CD$.	**Con.** $\angle 3 = \angle 6$.



Observe that the statements "$\angle 3 = \angle 6$" and "$AB \parallel CD$" of XIII are interchanged in XV.

When the hypothesis and conclusion of a theorem each consists of a *single* statement, the converse theorem is formed by interchanging the hypothesis and conclusion of the given theorem.

The converse of a given theorem is not always true.

Thus, *all right angles are equal.* The converse would be *all equal angles are right angles.* Evidently this is not true.

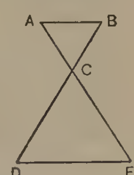
Ex. 114. Of what statement is Cor. 1 (§ 100) the converse? Cor. 2 (§ 101)? Cor. 3 (§ 102)?

Ex. 115. If two parallels are cut by a transversal, alternate exterior angles are equal.

Ex. 116. State the converse of Ex. 115. Is it a true statement?

Ex. 117. If two parallels are cut by a transversal, exterior angles on the same side of the transversal are supplementary.

Suggestion. The proof is like that for § 102.



Ex. 118. Prove the converse of Ex. 117.

Ex. 119. If $\triangle CDE$ at the right is isosceles, and $AB \parallel DE$, then $\angle A = \angle B$.

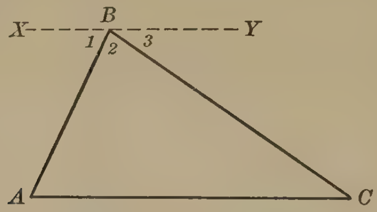
Ex. 120. Draw a figure like the one adjoining, having $AB \parallel DE$, and $AC = CE$. Prove $AB = DE$.

Ex. 121. Prove that a line through the vertex of an isosceles triangle, parallel to the base, bisects the exterior angles at the vertex of the triangle.

Note. The various theorems about angles made by a transversal of two parallels are found in Euclid. They were probably formulated by the Pythagoreans.

* PROPOSITION XVII. THEOREM

105. *The sum of the angles of any triangle is one straight angle.*



Hypothesis. $\triangle ABC$ is any triangle.

Conclusion. $\angle A + \angle B + \angle C = 1$ st. $\angle$.

Plan. Prove $\measuredangle A$, B, and C = $\measuredangle$ whose sum is a st. $\angle$.

Proof:

Statements	Reasons
1. Through B, draw $XY \parallel AC$.	1. § 98, p. 49.
2. $\therefore \angle A = \angle 1$.	2. Why?
3. $\angle B = \angle 2$.	3. Why?
4. $\angle C = \angle 3$.	4. Why?
5. $\angle 1 + \angle 2 + \angle 3 = 1$ st. $\angle$.	5. § 35, p. 13.
6. $\therefore \angle A + \angle B + \angle C = 1$ st. $\angle$.	6. Ax. 2, § 49, p. 18.

Note. This theorem is attributed to Eudemus, a pupil of Aristotle.

Ex. 122. If $\angle A = 70°$ and $\angle B = 35°$, how large is $\angle C$?

Ex. 123. How large is each angle of an equiangular triangle?

Ex. 124. How large is each of the other angles of an isosceles triangle, if the vertex angle is 30°?

Ex. 125. How large is each of the other angles of an isosceles triangle, if one of the base angles is 35°?

Ex. 126. What is the sum of the other two angles of a triangle when the third angle is a right angle?

Ex. 127. Two angles of a triangle are 50° and 66°, respectively. Find the angle made by the bisectors of these two angles.

Ex. 128. Prove Proposition XVII by using the figure of § 109.

Ex. 129. If two angles of a triangle are given, construct the third angle.

Ex. 130. Construct an angle of 60°; of 30°; of 120°.

106. A triangle is a **right triangle** when it has one right angle.

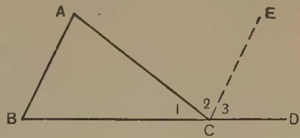
The **hypotenuse** of a right triangle is the side opposite the right angle; the **legs** of a right triangle are the two sides of the triangle including the right angle.

If the legs of a right triangle are equal, the triangle is called an **isosceles right triangle.**

Corollaries to Proposition XVII

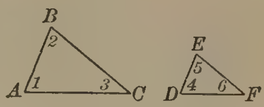
107. Cor. 1. *A triangle cannot have two right angles or two obtuse angles.*

108. Cor. 2. *The acute angles of a right triangle are complementary.*



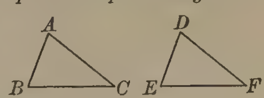
109. Cor. 3. *An exterior angle of a triangle equals the sum of the two remote interior angles.*

Prove $\angle DCA = \angle A + \angle B$.



110. Cor. 4. *If two angles of one triangle equal respectively two angles of another triangle, the third angles are equal.*

Hyp. $\angle 1 = \angle 4$, and $\angle 2 = \angle 5$.
Con. $\angle 3 = \angle 6$.



111. Cor. 5. *If two triangles have a side, the opposite angle, and another angle of the one equal respectively to a side, the opposite angle, and another angle of the other, the triangles are congruent.* (s. a. a. = s. a. a.)

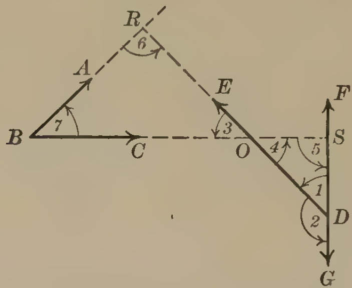
Hyp. $AB = DE$; $\angle C = \angle F$; $\angle B = \angle E$.
Con. $\triangle ABC \cong \triangle DEF$.

Suggestions. 1. Prove $\angle A = \angle D$.
2. Then prove $\triangle ABC \cong \triangle DEF$ by § 66.

Note. Additional Exercises 24–37, p. 274, can be studied now.

PROPOSITION XVIII. THEOREM

112. *If two angles have their sides respectively perpendicular, they are either equal or supplementary.*



Hypothesis. $\measuredangle CBA$, FDE, and EDG have $AB \perp DE$ and $BC \perp FG$.

Conclusion. $\angle CBA = \angle FDE$;
$\angle CBA$ is supplementary to $\angle EDG$.

Plan. Use § 110. (State it in full.)

Proof:

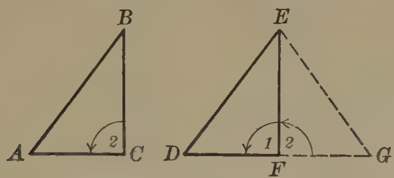
	Statements		Reasons
1.	Extend AB to meet DE at R. Extend BC to meet FG at S, crossing DE at O. In $\triangle RBO$ and $\triangle DOS$:	1.	$AB \perp DE$, by hypothesis. $BC \perp FG$, by hypothesis.
2.	$\angle 6 = \angle 5$;	2.	Why?
3.	$\angle 3 = \angle 4$;	3.	Why?
4.	$\therefore\ \angle 7 = \angle 1$.	4.	§ 110.
5.	$\angle 1$ is supp. of $\angle 2$.	5.	§ 40.
6.	$\therefore\ \angle 7$ is supp. of $\angle 2$.	6.	Ax. 2, § 49, p. 18.

Note. The angles are equal if both are acute, or obtuse; they are supplementary if one is acute and the other obtuse.

Or, if the angles are read in the counter-clockwise manner (see § 21, p. 9), the angles are equal if their initial sides are perpendicular, and they are supplementary if the initial side of one is perpendicular to the terminal side of the other.

* PROPOSITION XIX. THEOREM

113. *If two right triangles have the hypotenuse and a leg of one equal respectively to the hypotenuse and a leg of the other, the triangles are congruent.*



Hypothesis. In rt. $\triangle ABC$ and DEF, $\angle C$ and $\angle F$ are rt. $\angle$s;
$AB = DE$; $BC = EF$.

Conclusion. $\triangle ABC \cong \triangle DEF$.

Plan. Prove $\angle A = \angle D$; then use s. a. a. = s. a. a.

Proof: Statements	Reasons
1. Place $\triangle ABC$ so that BC coincides with EF, B on E, and A falls at G on the opposite side of EF from D.	1. § 60. $BC = EF$, by hypothesis.
2. $\angle 2 =$ rt. $\angle$; $\angle 1 =$ rt. $\angle$.	2. By hypothesis.
3. $\therefore \angle DFG =$ st. $\angle$.	3. A st. $\angle =$ sum of two rt. $\angle$s.
4. $\therefore DFG$ is a st. line.	4. § 41.
5. $\therefore EDFG$ is a $\triangle$.	5. DE, EG and DG are straight lines.
6. $ED = EG$.	6. EG or $AB = ED$, by hypothesis.
7. $\therefore \angle D = \angle G$, or $\angle D = \angle A$.	7. § 68.
8. $\therefore \triangle ABC \cong \triangle DEF$.	8. § 111. (Give full proof.)

Ex. **131.** If two right triangles have the hypotenuse and an acute angle of one equal respectively to the hypotenuse and an acute angle of the other, they are congruent. (Use § 111.)

Ex. **132.** If two right triangles have a leg and the opposite acute angle of one equal respectively to a leg and the opposite acute angle of the other, they are congruent.

114. Summarizing review. Before proceeding, every pupil should understand and be able to state in full the theorems, definitions and postulates called for in the following exercises, and should be able to prove at least all of these theorems which are marked in the text with a star.

Ex. 133. State four theorems by which two triangles are proved congruent. (§§ 62, 66, 73, and 111.)

Ex. 134. State three theorems by which two right triangles are proved congruent. (§ 113; Ex. 131, and Ex. 132.)

Ex. 135. State five theorems by which two lines are proved parallel. (§§ 90, 92, 95, 96, and 97.)

Ex. 136. State five theorems about pairs of angles when two parallels are cut by a transversal. (§§ 99, 100, 102; Ex. 115 and 117.)

Ex. 137. State eleven theorems by which two angles can be proved equal. (§§ 27, 38, 42, 54, 65, 68, 99, 100, 103, 110, 112.)

Ex. 138. State four theorems by which two angles can be proved supplementary. (§§ 30, 40, 102, and Ex. 117.)

Ex. 139. State two theorems about an exterior angle of a triangle. (§§ 86 and 109.)

Ex. 140. Give the definitions of the following terms:

(*a*) Angle.
(*b*) Straight angle.
(*c*) Acute angle.
(*d*) Obtuse angle.
(*e*) Isosceles triangle.
(*f*) Equiangular triangle.
(*g*) Equilateral triangle.
(*h*) Perpendicular-bisector.
(*i*) Median.
(*j*) Altitude.
(*k*) Distance from a point to a line.
(*l*) Exterior angle of a triangle.
(*m*) Parallel lines.
(*n*) Right triangle.
(*o*) Hypotenuse.
(*p*) Isosceles right triangle.
(*q*) Congruent triangles.
(*r*) Scalene triangle.

Ex. 141. State the following postulates:

(*a*) The postulate of motion.
(*b*) The postulate of parallels.
(*c*) The postulates concerning the straight line.
(*d*) The three postulates concerning a straight line-segment.
(*e*) The postulate concerning the circle.
(*f*) The two postulates concerning angles.

Ex. 142. State the two fundamental plans of demonstration learned thus far. (§§ 65, 94.)

115. Demonstrating unproved theorems is one of the chief goals in studying geometry. Some success can be achieved by summarizing known facts, from time to time, as in § 114, by planning a demonstration systematically, and by persevering.

1. *Read the theorem carefully,* making certain that each word is thoroughly understood.

2. *Draw the figure carefully,* constructing it when possible.

Make the figure general. Thus, if it is a triangle, do not draw a right triangle or an isosceles triangle unless told to do so.

3. *Decide upon the hypothesis and conclusion.*

(*a*) Remember that the hypothesis states the facts about the figure which are assumed, and that the conclusion states the facts which are to be proved. One help in making the hypothesis is to describe *at once* in the hypothesis each point which is placed in a special position (as a mid-point), or each line (as an altitude), or segments which are made equal (as the sides of an isosceles triangle).

(*b*) If the theorem is stated in the "if . . ., then . . ." form, the hypothesis and conclusion are evident at once. (See § 53.)

(*c*) If the theorem is not stated in the "if . . ., then . . ." form the subject of the sentence with its modifiers gives the hypothesis, and the predicate gives the conclusion.

Thus, in "the base angles of an isosceles triangle are equal," "the base angles of an isosceles triangle" must be described in the hypothesis, and "are equal" indicates the conclusion.

4. *Make a plan for the demonstration.*

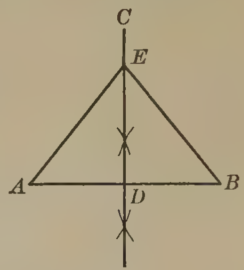
(*a*) Ask "*what does the conclusion mean*" and, when you know what it means, ask "*how can I prove the conclusion.*" Facts like those summarized in § 114 will suggest a plan, as a rule.

(*b*) After you have decided upon a plan, ask "*what do I know about the figure which will help me to carry out my plan.*" Use, first, the facts given in the hypothesis, and next, any facts derived from these or from the figure.

Practice in planning a demonstration will be given in the next four pages.

* PROPOSITION XX. THEOREM

116. I. *Any point in the perpendicular-bisector of a segment is equidistant from the ends of the segment.*



Hypothesis. $CD \perp AB$; $AD = DB$;
E is any point in CD.

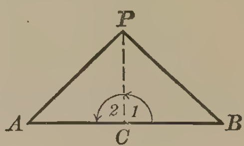
Conclusion. $EA = EB$.

(*Mentally.* How can I prove $EA = EB$?)

Plan. Try to prove EA and EB corres. sides of cong. $\triangle$s.

(Proof to be given by the pupil.)

II. (Converse.) *Any point equidistant from the ends of a segment lies in the perpendicular-bisector of the segment.*



Hypothesis. AB is a st. line segment.
$PA = PB$.

Conclusion. P lies in the perpendicular-bisector of AB.

(*Mentally.* 1. What does $\perp$-bis. of AB mean?
Answer. A line $\perp$ AB at its mid-point.
2. How can I prove P is on such a line?)

Plan. Construct $PC \perp AB$ from P.
Try to prove AC and CB corres. sides of cong. $\triangle$s.

(Proof to be given by the pupil.)

Ex. 143. Give the following proof for Part II of § 116.

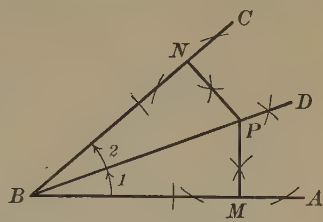
(*Mentally.* 1. What does $\perp$-bis. of AB mean?
Answer. A line $\perp$ AB at its mid-point.
2. How can I prove P lies on such a line?)

Plan. 1. Draw a line from P to C, the mid-point of AB.
2. Try to prove $PC \perp AB$, by proving $\angle 1 = \angle 2$. Use Plan I.

Note for Teacher. Locus of points, postponed in this text to pages 128–130, *can be* studied now if desired.

* PROPOSITION XXI. THEOREM

117. I. *Any point in the bisector of an angle is equidistant from the sides of the angle.*



Hypothesis. BD bisects $\angle ABC$; P is in BD; $PM \perp AB$; $PN \perp BC$.

Conclusion. $PM = PN$.

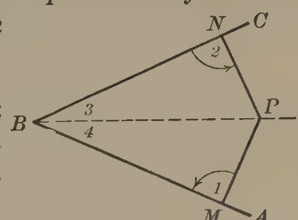
(*Mentally.* How can I prove $PM = PN$?)

Plan. Try to prove PM and PN corres. sides of cong. ⚠.

Proof:

STATEMENTS	REASONS
In $\triangle PBN$ and $\triangle PBM$:	
1. $\angle BNP$ and $\angle PMB$ are rt. ∡.	1. $PN \perp BC$, and $PM \perp AB$, by hypothesis.
2. $\therefore \angle BNP = \angle PMB$.	2. Why?
3. $\angle PBN = \angle MBP$.	3. BD bisects $\angle ABC$, by hyp.
4. $BP = BP$.	4. Why?
5. $\therefore \triangle PBN \cong \triangle PBM$.	5. § 111.
6. $\therefore PN = PM$.	6. Why?

II. (Converse.) *Any point equidistant from the sides of an angle lies in the bisector of the angle.*



Hypothesis. P lies within $\angle ABC$; $PM \perp AB$; $PN \perp BC$; $PM = PN$.

Conclusion. P lies in the bisector of $\angle ABC$.

(*Mentally.* How can I prove P lies in the bis. of $\angle ABC$?)

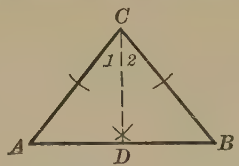
Plan. Draw PB. Try to prove $\angle 3 = \angle 4$ by proving them corr. angles of cong. ⚠.

(Proof to be given by the pupil.)

Note. Additional Exercises 38 to 41, p. 275, can be studied now.

* PROPOSITION XXII. THEOREM

118. *If two angles of a triangle are equal, the sides opposite are equal, and the triangle is isosceles.*



Hypothesis. In $\triangle ABC$, $\angle A = \angle B$.

Conclusion. $AC = BC$.

Plan. Try to prove AC and BC corres. sides of cong. $\triangle$.

Proof:

STATEMENTS	REASONS
1. Construct CD, bisecting $\angle ACB$, meeting AB at D.	1. An angle can be bisected, § 74.
In $\triangle ADC$ and $\triangle BDC$:	
2. $\angle A = \angle B$.	2. Why?
3. $CD = CD$.	3. Why?
4. $\angle ACD = \angle DCB$.	4. Construction in Step 1.

(Complete the proof, using § 111 and § 65.)

119. Cor. *If a triangle is equiangular, it is also equilateral.*

Ex. 144. Prove that the bisector of the vertical angle of an isosceles triangle is perpendicular to and bisects the base.

Ex. 145. Prove that the altitude to the base of an isosceles triangle is also the median to the base and bisects the vertical angle.

Ex. 146. Prove that the altitudes drawn to the equal sides of an isosceles triangle are equal.

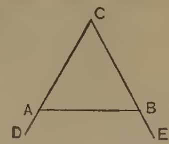
Ex. 147. Prove that the medians drawn to the equal sides of an isosceles triangle are equal.

Ex. 148. If one angle of an isosceles triangle is 60°, the triangle is equilateral.

Ex. 149. If the perpendiculars drawn from the mid-point of one side of a triangle to the other two sides are equal, the triangle is isosceles.

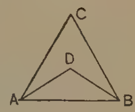
Ex. 150. Prove that the perpendiculars drawn from the mid-point of the base of an isosceles triangle to the sides of the triangle are equal.

Ex. 151. If the equal sides of an isosceles triangle be extended beyond the base, the exterior angles so formed are equal.



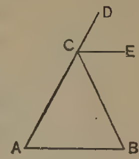
Ex. 152. If the altitudes drawn to two sides of a triangle are equal, the triangle is isosceles.

Ex. 153. The bisectors of the equal angles of an isosceles triangle form with the base another isosceles triangle.



Ex. 154. If the bisector of the exterior angle at one vertex of a triangle is parallel to the side joining the other two vertices, the triangle is isosceles.

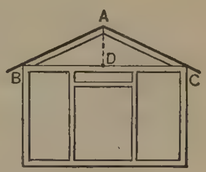
Ex. 155. Prove that the bisector of the exterior angle at the vertex of an isosceles triangle is parallel to the base of the triangle.



Suggestion. Compare $\angle BCD$ with $\angle A + \angle B$ (§ 109).

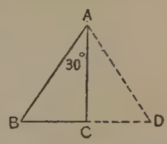
Ex. 156. In the gable in the front of a garage, the two boards whose upper edges are AB and AC are of equal length and meet at a point A on a line AD which is perpendicular to BC.

If $\angle ACD = 30°$, how large are $\angle ABD$, $\angle CAD$, and $\angle BAD$?



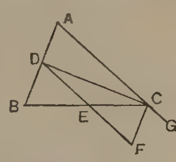
Ex. 157. If one acute angle of a right triangle is 30°, the side opposite is one-half the hypotenuse.

Suggestion. Extend BC to D, making CD equal to BC. Prove $\triangle ABD$ is equilateral.



Note. This is a very important exercise. Pupils should endeavor to remember it, as it will be required in the proofs of certain exercises in geometry.

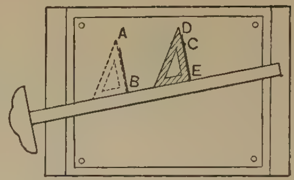
Ex. 158. If CD is the bisector of $\angle C$ of $\triangle ABC$, and DF be drawn parallel to AC meeting BC at E and the bisector of the angle exterior to C at F, prove $DE = EF$.



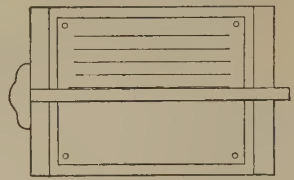
Suggestion. Compare DE and EF with EC.

Note. Additional Exercises 42–51, p. 275, can be studied now.

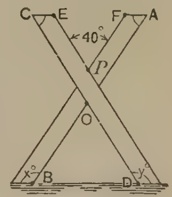
Practical Applications



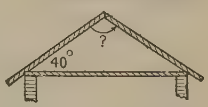
Ex. 159. The figure adjoining shows how a draughtsman draws a line through C parallel to AB. Why is DE parallel to AB?



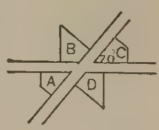
Ex. 160. The adjoining figure shows how a draughtsman draws parallel lines by means of his T-square. Why are the lines parallel?



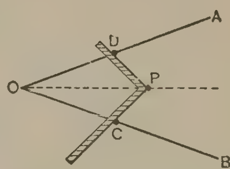
Ex. 161. A boy wishes to make a saw-buck. Assume that $BO = OD$ and that $\angle EPF = 40°$. Determine the angles at B and D so that the pieces AB and CD will stand firmly upon the ground. Determine the angles at C and A so that CE and FA will be parallel to the ground line.



Ex. 162. The rafters of a "saddle roof" make an angle of 40° with a level line. What angle do the rafters form at the ridge?



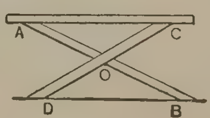
Ex. 163. Two streets cross as in the adjoining figure. If the lot lines at corner C make an angle of 70°, determine the number of degrees in the angle formed at each of the other corners.



Ex. 164. Books for carpenters give the following method of bisecting an angle by means of the "square" alone.

(*a*) Make OD and OC of equal length. Place the square so that $DP = CP$. Then OP bisects $\angle AOB$. Prove that the method is correct.

(*b*) Is it necessary to use a "square"; that is, a tool in which $\angle DPC$ is a right angle?

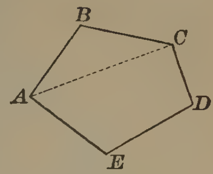


Ex. 165. An ironing board is supported as shown in the adjoining figure. If $AO = OB$ and $DO = OC$, prove that AC is always parallel to the floor DB.

POLYGONS

120. A **polygon** is a *closed* (§ 15) broken line in a plane.

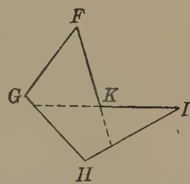
Points A, B, C, etc., are the *vertices* of the polygon; ∡ A, B, C, etc., are the *angles*; AB, BC, CD, etc., are the *sides*; the sum of the lengths of the sides is the *perimeter* of the polygon; a line joining any two non-consecutive vertices is a *diagonal* of the polygon; as AC.



121. A polygon is **convex** if no side, when extended, will pass through the interior of the polygon; as $ABCDE$ of § 120.

A convex polygon incloses a portion of the plane called the **interior of the polygon.**

In higher geometry, *concave polygons* are studied, as $FGHIK$. Only convex polygons are studied in this text.



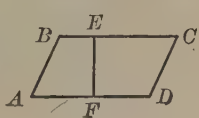
122. An **equilateral polygon** is one whose sides are all equal. An **equiangular polygon** is one whose angles are all equal.

123. The principal polygons are named as follows:

No. of Sides	Name of the Polygon	No. of Sides	Name of the Polygon
3	Triangle	7	Heptagon
4	Quadrilateral	8	Octagon
5	Pentagon	10	Decagon
6	Hexagon	n	n-gon

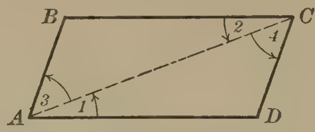
124. A **parallelogram** (▱) is a quadrilateral whose opposite sides are parallel; as $ABCD$.

A pair of parallel sides are called *bases*; the perpendicular distance between them is called the *altitude*, as EF.



PROPOSITION XXIII. THEOREM

125. *A diagonal of a parallelogram divides it into two congruent triangles.*



Hypothesis. $ABCD$ is a parallelogram. AC is a diagonal.

Conclusion. $\triangle ABC \cong \triangle ACD$.

(Proof to be given by the pupil.)

Suggestions. 1. Since $AD \parallel BC$, compare $\angle 1$ and $\angle 2$.
2. Compare $\angle 3$ and $\angle 4$. What are the parallels?

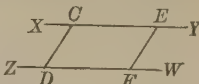
126. Cor. 1. *The opposite sides of a parallelogram are equal.*

127. Cor. 2. *The opposite angles of a parallelogram are equal.*

128. Cor. 3. *Two consecutive angles of a parallelogram are supplementary.*

Suggestion. Prove $\angle A + \angle B = 1$ st. $\angle$, in the figure for § 125.

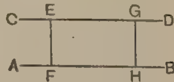
129. Cor. 4. *Segments of parallels included between parallels are equal.*



Hyp. $XY \parallel ZW$; $CD \parallel EF$.

Con. $CD \doteq EF$.

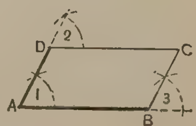
130. Cor. 5. *Two parallels are everywhere equidistant.*



Hyp. $CD \parallel AB$.
EF and GH are any two $\perp$s to CD and AB.

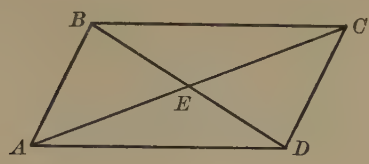
Con. $EF = GH$.

Ex. 166. Construct a ▱ $ABCD$, making $AD = 2$ in., $AB = 3$ in., and $A = 60°$. After you have constructed the figure, compare the opposite sides by means of your dividers; compare the opposite $\angle$s; compare the consecutive angles.



* PROPOSITION XXIV. THEOREM

131. *The diagonals of a parallelogram bisect each other.*



Hypothesis. $ABCD$ is a ▱.

Diagonals AC and BD intersect at E.

Conclusion. $AE = EC$; $BE = ED$.

Plan. Try to prove AE and EC corres. sides of cong. △.

(Proof to be given by the pupil.)

132. The point of intersection of the diagonals of a parallelogram is the **center** of the parallelogram.

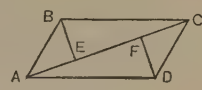
Ex. 167. (*a*) If one angle of a parallelogram is 100°, how large is each of the other angles?

(*b*) If one angle of a parallelogram is a right angle, the others are also.

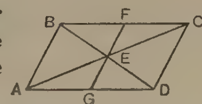
Ex. 168. If two adjacent sides of a parallelogram are equal, all its sides are equal.

Ex. 169. If perpendiculars BE and DF are drawn to the diagonal AC of a parallelogram $ABCD$, then $BE = DF$.

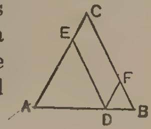
(Construct the figure with ruler and compasses.)



Ex. 170. If a line be drawn through the center of a parallelogram and terminated by two opposite sides of the parallelogram, it is bisected by the center.

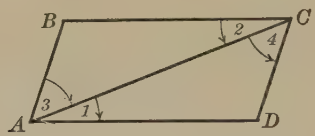


Ex. 171. If from any point in the base of an isosceles triangle parallels to the equal sides be drawn: (*a*) a parallelogram is formed, and (*b*) the perimeter of the parallelogram formed is equal to the sum of the equal sides of the triangle.



* PROPOSITION XXV. THEOREM

133. *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*



Hypothesis. $BC \parallel AD$; $BC = AD$.

Conclusion. $ABCD$ is a $\square$.

(*Mentally.* 1. What is a parallelogram?
Answer. A figure whose opposite sides are parallel.
2. Since $BC \parallel AD$, I must prove $AB \parallel CD$. How?)

Plan. Try to prove $\angle 3 = \angle 4$, by proving $\triangle ABC \cong \triangle ACD$.

Proof:

STATEMENTS	REASONS
1. $\triangle ABC \cong \triangle ACD$.	1. Give the full proof.
2. $\therefore \angle 3 = \angle 4$.	2. Why?
3. $\therefore AB \parallel CD$.	3. Why?
4. And $BC \parallel AD$.	4. Why?
5. $\therefore ABCD$ is a $\square$.	5. Why?

Note. When giving the proof, omit the portion above marked "Mentally."

Ex. 172. The line joining the mid-points of two opposite sides of a parallelogram is parallel to the other two sides.

(Prove $AEFD$ is a $\square$ and therefore $EF \parallel AD$.)

Ex. 173. If $ABCD$ is a parallelogram, and E and F are the mid-points of AB and CD respectively, then $AECF$ is also a parallelogram.

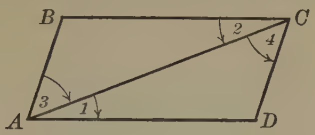
Ex. 174. If $ABCD$ is a $\square$; if X and Y divide AB so that $AX = XY = YB$; if Z and W divide CD, so that $CZ = ZW = WD$; then $YZ \parallel XW$.

Ex. 175. In the figure of Ex. 174, prove $DX \parallel BZ$.

Ex. 176. Prove that two straight lines are parallel if any two points of one are equidistant from the other. (Recall § 83.)

* PROPOSITION XXVI. THEOREM

134. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*



Hypothesis. $AB = CD$; $BC = AD$.

Conclusion. $ABCD$ is a parallelogram.

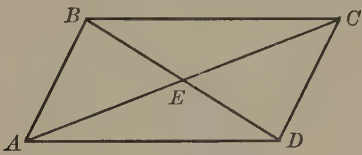
Plan. Prove $AB \parallel CD$, by proving $\angle 3 = \angle 4$.
Prove $BC \parallel AD$, by proving $\angle 2 = \angle 1$.

Proof:

STATEMENTS	REASONS
1. $\triangle ABC \cong \triangle ACD$.	1. Give the full proof.
2. $\therefore \angle 3 = \angle 4$.	2. Why?
3. $\therefore AB \parallel CD$.	3. Why?
4. Also $BC \parallel AD$.	4. Give the full proof.
5. $\therefore ABCD$ is a $\square$.	5. Why?

PROPOSITION XXVII. THEOREM

135. *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*



Hypothesis. In quadrilateral $ABCD$,
$AE = EC$, and $BE = ED$.

Conclusion. $ABCD$ is a $\square$.

(The Plan and Proof are to be given by the pupil.)

Note. Additional Exercises 52–55, p. 276, can be studied now.

SPECIAL PARALLELOGRAMS

136. A **rectangle** is a parallelogram one of whose angles is a right angle. It can be proved and it is important to remember that *all the angles of a rectangle are right angles.*

Note. Since a rectangle is a special parallelogram, every theorem true about parallelograms is true about rectangles. Thus, the diagonals of a rectangle bisect each other. On the other hand, theorems true about a rectangle are not necessarily true about a parallelogram, since a rectangle is a *special* parallelogram.

Ex. **177.** State some other properties of a rectangle which follow at once from properties of a parallelogram. (See §§ 125–132.)

Ex. **178.** Construct a rectangle whose sides are 1.5 in. and 2 in. respectively. Draw and measure its diagonals.

Ex. **179.** Prove that the diagonals of a rectangle are equal.

Ex. **180.** Prove that a quadrilateral whose angles are all right angles is a rectangle.

Ex. **181.** Prove that a parallelogram whose diagonals are equal is a rectangle.

Plan. Try to prove one of its $\angle$s is a right angle. To do this, prove $\angle BAD = \angle ADC$; then use § 102.

137. A **rhombus** is a parallelogram having two adjacent sides equal. It *can be proved,* and it is important to remember, that *all the sides of a rhombus are equal;* also it is *usually* implied that the angles are not right angles. (See § 138.)

Ex. **182.** State properties of a rhombus which are evident at once because the rhombus is a special parallelogram. (See Note, § 136.)

Ex. **183.** Prove that the diagonals of a rhombus are perpendicular to each other.

Ex. **184.** Prove that the diagonals of a rhombus bisect the angles.

Ex. **185.** Construct a rhombus whose sides are each 3 in. and whose acute angles are each 45°. Draw and measure its diagonals.

Ex. **186.** Construct a rhombus, having given one side and one diagonal.

Ex. **187.** Prove that the two altitudes of a rhombus are equal.

138. A **square** is a parallelogram having two adjacent sides equal and one angle a right angle. It *can be proved,* and it is important to remember, that *all the angles of a square are right angles and all the sides are equal.*

Many artistic designs are made on a network of squares, as illustrated below.

CORNER AND BORDER FOUR BORDER DESIGNS

Note. The square is a special rectangle and also a special rhombus. Hence every theorem true about a rectangle or a rhombus is true about a square. (See Note, § 136.)

Ex. 188. What facts about the square may be inferred from known facts about the parallelogram, the rectangle, and the rhombus?

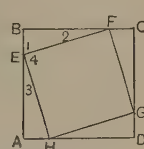
Ex. 189. How large are the angles into which a diagonal of a square divides its angles?

Ex. 190. Construct a square whose diagonals shall be 2 in. in length.

Ex. 191. Prove that the lines drawn from the ends of one side of a square to the mid-points of the two adjacent sides are equal.

Ex. 192. Prove that if the diagonals of a quadrilateral are perpendicular to and bisect each other, the figure is a rhombus.

Ex. 193. If E, F, G, and H are points on the sides, AB, BC, CD, and AD respectively of square $ABCD$, such that $AE = BF = CG = DH$, prove that $EFGH$ is a square.



Suggestions. 1. Try to prove $EFGH$ is a ▱, having two adj. sides equal, and having one $\angle$ ($\angle 4$) a right angle.

2. To prove $\angle 4$ a right angle:

(*a*) $\angle 1 + \angle 2 = ?$ (*b*) Does $\angle 3 = \angle 2$?

(*c*) $\angle 1 + \angle 3 + \angle 4 = ?$ (*d*) $\therefore\ \angle 4 = ?$

Note. Supplementary Exercises 56–61, p. 276, can be studied now.

139. A **trapezoid** is a quadrilateral which has one and only one pair of parallel sides; AB and CD are called the non-parallel sides.

The parallel sides of a trapezoid are called the **bases.**

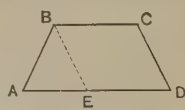
The perpendicular distance between the bases is called the **altitude.**

The line joining the mid-points of the non-parallel sides is called the **median** of the trapezoid.

140. An **isosceles trapezoid** is a trapezoid whose non-parallel sides are equal.

Ex. **194.** If the angles at the ends of one base of a trapezoid are equal, the angles at the ends of the other base are also equal.

Ex. **195.** If a trapezoid is isosceles, the lower base angles are equal. (If $AB = CD$, prove $\angle A = \angle D$. Draw $BE \parallel CD$. Compare $\angle AEB$ with $\angle D$ and $\angle A$.)



Ex. **196.** If one pair of base angles of a trapezoid are equal, the trapezoid is isosceles.

Ex. **197.** Prove that the diagonals of an isosceles trapezoid are equal.

Ex. **198.** Prove that the opposite angles of an isosceles trapezoid are supplementary.

Note. Additional Exercise 62, p. 277, can be studied now.

REVIEW OF QUADRILATERALS

Ex. **199.** Define: (*a*) polygon; (*b*) quadrilateral; (*c*) parallelogram; (*d*) rectangle; (*e*) rhombus; (*f*) square; (*g*) trapezoid; (*h*) isosceles trapezoid; (*i*) median of trapezoid.

Ex. **200.** State four theorems by which to prove a quadrilateral is a parallelogram.

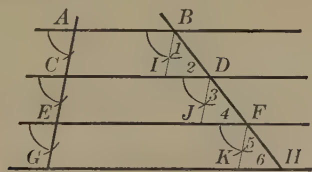
Ex. **201.** If E, F, G, and H are mid-points of sides AB, BC, CD, and AD respectively of parallelogram $ABCD$, then $EFGH$ is a parallelogram.

Ex. **202.** If diagonal AC of $\square$ $ABCD$ is trisected at X and Y, so that $AX = XY = YC$, then $BXDY$ is also a parallelogram.

Ex. **203.** If diagonal AC of a square $ABCD$ is trisected at X and Y, so that $AX = XY = YC$, then $BXDY$ is a rhombus.

* PROPOSITION XXVIII. THEOREM

141. *If three or more parallels intercept equal lengths on one transversal, they intercept equal lengths on all transversals.*



Hypothesis. $AB \parallel CD \parallel EF \parallel GH$.
AG cuts the $\parallel$s at A, C, E, and G.
BH cuts the $\parallel$s at B, D, F, and H.
$AC = CE = EG$.

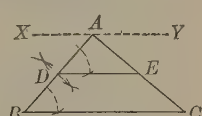
Conclusion. $BD = DF = FH$.

Plan. Try to prove BD, DF, and FH corres. sides of cong. $\triangle$s.

Proof: STATEMENTS	REASONS
1. Draw BI, DJ, $FK \parallel AG$.	1. § 98.
2. $\therefore BI \parallel DJ \parallel FK$.	2. § 90.
3. $BI = AC$; $DJ = CE$; $FK = EG$.	3. Why?
4. But $AC = CE = EG$.	4. Why?
5. $\therefore BI = DJ = FK$.	5. Why?

[Complete the proof by proving $\triangle$s BDI, DJF, and FHK are congruent, and then proving that $BD = DF = FH$.]

142. Cor. 1. *If a line bisects one side of a triangle, and is parallel to a second side, it bisects the third side also.*

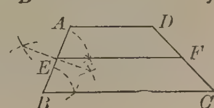


Hyp. D is on AB of $\triangle ABC$;
$AD = DB$; $DE \parallel BC$.

Con. $AE = EC$.

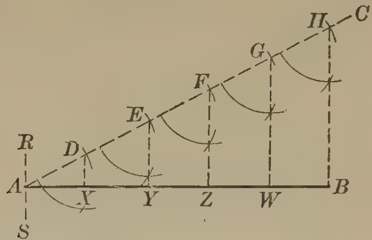
Proof. 1. Assume $XAY \parallel BC$.

143. Cor. 2. *If a line is parallel to the bases of a trapezoid and bisects one of the non-parallel sides, it bisects the other also.*



PROPOSITION XXIX. PROBLEM

144. *Divide a given segment into any number of equal parts.*



Given segment AB.

Required to divide AB into five equal parts.

Construction. 1. Draw line AC, making a convenient $\angle$ with AB.

2. Upon AC, lay off $AD = DE = EF = FG = GH$.

3. Draw HB.

4. Through D, E, F, G, and H, draw lines parallel to HB, meeting AB at X, Y, Z, and W.

Statement. $AX = XY = YZ = ZW = WB$.

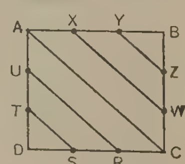
Plan. Use § 141.

Proof: Statements	Reasons
1. Assume RS through A parallel to HB.	1. Possible.
2. $\therefore RS \parallel DX \parallel EY \parallel FZ \parallel GW \parallel HB$.	2. Why?
3. $\therefore AX = XY = YZ = ZW = WB$.	3. Why?

Ex. 204. Draw a segment about 4 inches long. Divide it into six equal parts.

Ex. 205. (*a*) Construct an isosceles triangle ABC, having $BC = 2$ in., and $AB = AC = 3$ in. Let X and Y trisect at BC.

(*b*) Prove $AX = AY$.



Ex. 206. Let $ABCD$ be a square. Let X and Y trisect AB; Z and W trisect BC; R and S trisect CD; and T and U trisect DA.

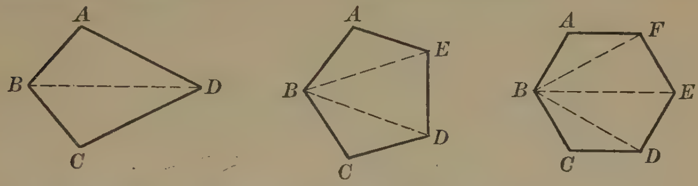
(*a*) Prove $YZ \parallel XW \parallel AC \parallel RU \parallel ST$.

(*b*) Prove $YZRU$ is an isosceles trapezoid.

Note. Additional Exercises 63–65, p. 277, can be studied now.

PROPOSITION XXX. THEOREM

145. *The sum of the interior angles of a polygon having n sides is $(n-2)$ straight angles.*



Hypothesis. Assume a polygon of n sides.

Conclusion. The sum of its int. $\angle s = (n-2)$ st. $\angle s$.

Proof. 1. Draw diagonals from B to each of the other vertices.

2. Each side of the polygon, excepting AB and BC, becomes the base of a triangle whose vertex is at B. Hence there are $(n-2)$ $\triangle s$ formed.

(When n is 4, there are 2 $\triangle s$; when n is 5, there are 3 $\triangle s$; etc.)

3. The sum of the int. $\angle s$ of each $\triangle$ is 1 st. $\angle$. Why?

4. $\therefore$ the sum of the int. $\angle s$ of the $(n-2)$ $\triangle s$ is $(n-2)$ st. $\angle s$.

5. But the sum of the int. $\angle s$ of the $\triangle s =$ the sum of the int. $\angle s$ of the polygon.

6. $\therefore$ the sum of the int. $\angle s$ of the polygon is $(n-2)$ st. $\angle s$.

Note. Since the sum of the $\angle s$ is $(n-2)$ st. $\angle s$, it also equals $(n-2) \cdot 2$ rt. $\angle s$, or $(n-2) \cdot 180°$.

Ex. 207. Express in straight angles, in right angles, and in degrees the sum of the angles of a polygon having:

(*a*) four sides; (*b*) five sides; (*c*) six sides; (*d*) eight sides.

Ex. 208. How many degrees are there in each angle of an equiangular: (*a*) quadrilateral? (*b*) pentagon? (*c*) hexagon? (*d*) octagon?

Ex. 209. If two angles of a quadrilateral are supplementary, then the other two are also.

Ex. 210. How many sides has a polygon the sum of whose angles is 16 right angles? 7 straight angles? 1620 degrees?

INEQUALITIES

146. The symbol for "less than" is $<$; for "greater than" is $>$.

147. Order of inequalities. $a < b$ and $c < d$ are two inequalities of the same order. $m < n$ and $x > y$ are two inequalities of opposite orders.

148. Axioms for combining inequalities.

Ax. 9. *If equals be added to unequals, the sums are unequal in the same order.*

Ax. 10. *If equals be subtracted from unequals, the differences are unequal in the same order.*

Thus, if $a > b$, then $a - c > b - c$.

Ax. 11. *If $a > b$ and $b > c$, then $a > c$.*

Ax. 12. *If unequals be added to unequals in the same order, the sums are unequal in the same order.*

Thus, if $a < b$, and $c < d$, then $a + c < b + d$.

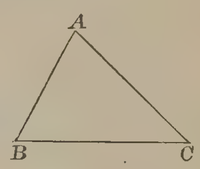
Ax. 13. *If unequals be subtracted from equals or from unequals of opposite order, the differences are unequal and of order opposite to that of the subtrahend.*

Thus, if $a > b$, and $c < d$, then $a - c > b - d$.

Arithmetical Example. Since $12 > 7$ and $3 < 5$, then $12 - 3$ should be greater than $7 - 5$. Is it?

149. Fundamental inequalities for segments.

(*a*) *Any side of a triangle is less than the sum of the other two sides.*



This follows from Post. 3, § 50.
Thus $BC < AB + AC$.

(*b*) *Any side of a triangle is greater than the difference of the other two sides.*

Thus, $BC > AC - AB$.

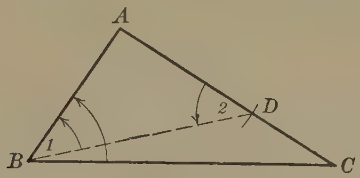
For, from (*a*) $BC + AB > AC$. Subtracting AB from both members of the inequality, $BC > AC - AB$, by Ax. 10, § 148.

150. Fundamental inequality for angles.

An exterior angle of a triangle is greater than either remote interior angle of the triangle. (§ 86.)

PROPOSITION XXXI. THEOREM

151. *If two sides of a triangle are unequal, the angles opposite are unequal, the angle opposite the greater side being the greater.*



Hypothesis. In $\triangle ABC$, $AC > AB$.

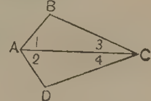
Conclusion. $\angle B > \angle C$.

Plan. Prove $\angle B >$ some $\angle$ known to be $> \angle C$.

Proof:

STATEMENTS	REASONS
1. On AC take $AD = AB$.	**1.** Since $AC > AB$, by hyp.
2. Draw BD.	**2.** § 5 (a).
3. $\therefore \angle 1 = \angle 2$.	**3.** Why?
4. $\angle 2 > \angle C$.	**4.** § 86.
5. $\therefore \angle 1 > \angle C$.	**5.** Why?
6. $\angle ABC > \angle 1$.	**6.** Ax. 8, § 49.
7. $\therefore \angle ABC > \angle C$.	**7.** Ax. 11, § 148.

Ex. 211. If a triangle is scalene, all its angles are unequal.

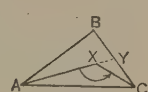


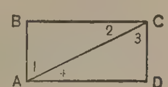
Ex. 212. If $AB = AD$, $BC = DC$, and $BC > AB$, prove $\angle BAD > \angle BCD$.

Ex. 213. In the adjoining figure, prove

$$\angle AXC > \angle ABC.$$

Suggestions. 1. Compare $\angle AXC$ with $\angle AYC$.
2. Compare $\angle AYC$ with $\angle ABC$.
3. Use Ax. 11, § 146.

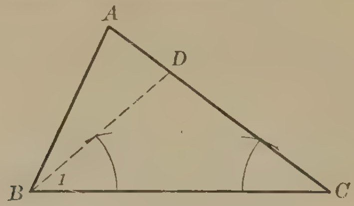




Ex. 214. In rectangle $ABCD$, $BC > AB$. Prove that diagonal AC does not bisect $\angle$s A and C.

PROPOSITION XXXII. THEOREM

152. *If two angles of a triangle are unequal, the sides opposite are unequal, the side opposite the greater angle being the greater.*



Hypothesis. In $\triangle ABC$, $\angle B > \angle C$.

Conclusion. $AC > AB$.

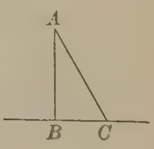
Plan. Prove $AB <$ something which equals AC.

Proof:

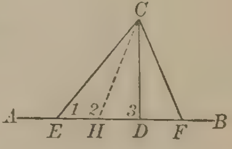
STATEMENTS	REASONS
1. In $\angle B$, construct $\angle 1 = \angle C$.	1. Possible, since $\angle B > \angle C$.
2. $\therefore BD = DC$.	2. Why?
3. $AB < AD + BD$.	3. Why?
4. $\therefore AB < AD + DC$.	4. Why?
5. $\therefore AB < AC$.	5. Since $AC = AD + DC$.

153. Cor. 1. *The hypotenuse of a right triangle is greater than either leg of the triangle.*

154. Cor. 2. *The perpendicular from a point to a line is the shortest segment from the point to the line.*



Ex. 215. If two segments, drawn from a point in a perpendicular to a line, cut off unequal distances from the foot of the perpendicular, the one cutting off the greater distance is the greater.



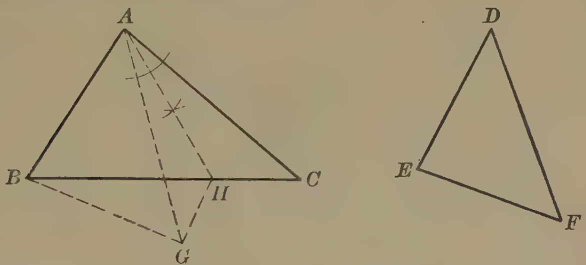
Hyp. $CD \perp AB$; $ED > DF$.

Con. $CE > CF$.

Suggestions. 1. Take $DH = DF$. Draw CH. Prove $CH = CF$. 2. Prove $\angle 2 > \angle 1$, by comparing each with $\angle 3$.

PROPOSITION XXXIII. THEOREM

155. *If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*



Hypothesis. In $\triangle ABC$ and $\triangle DEF$:

$AB = DE$; $AC = DF$; $\angle BAC > \angle D$.

Conclusion. $BC > EF$.

Proof:

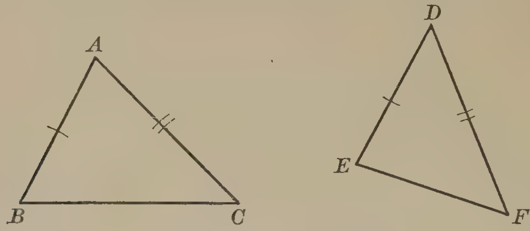
STATEMENTS	REASONS
1. Place $\triangle DEF$ in the position ABG, DE coinciding with AB.	1. § 60. $DE = AB$, by hyp.
2. DF falls at AG, inside $\angle BAC$.	2. $\angle BAC > \angle EDF$, by hyp.
3. Construct AH, bisecting $\angle GAC$, meeting BC at H. Draw GH.	3. § 74.
4. $\triangle GAH \cong \triangle AHC$.	4. Give the full proof.
5. $\therefore HG = HC$.	5. Why?
6. In $\triangle BGH$, $BH + HG > BG$.	6. Why?
7. $\therefore BH + HC > BG$.	7. Why?
8. $\therefore BC > EF$.	8. Ax. 2, § 49.

Note. This proof assumes that G falls below BC. If G falls on BC, then EF is at once less than BC. If G falls above BC, then a proof similar to the one above may be given.

Ex. 216. If O is any point in the base BC of isosceles triangle ABC, then AO is less than AC. (Prove $\angle AOC > \angle ACO$.)

PROPOSITION XXXIV. THEOREM

156. *If two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.*



Hypothesis. In $\triangle ABC$ and $\triangle DEF$:
$AB = DE$; $AC = DF$; $BC > EF$.

Conclusion. $\angle A > \angle D$.

Plan. Use the indirect method of proof.

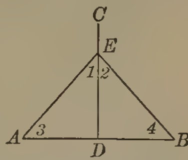
Proof:

STATEMENTS	REASONS
1. Suppose $\angle A$ not $> \angle D$.	1. $\angle A$ must be $> \angle D$ or not $> \angle D$.
2. If $\angle A = \angle D$, then $\triangle ABC \cong \triangle DEF$.	2. One of the possibilities. Give the full proof.
3. Then $BC = EF$.	3. Why?
4. But $BC > EF$.	4. By the hypothesis.
5. $\therefore \angle A$ cannot equal $\angle D$.	5. Supposing $\angle A = \angle D$ leads to a contradiction. See § 93.
6. If $\angle A < \angle D$,	6. A second possibility.
7. then $BC < EF$.	7. § 155. Give the full proof.
8. But $BC > EF$.	8. By the hypothesis.
9. $\therefore \angle A$ cannot be $< \angle D$.	9. Supposing $\angle A < \angle D$ leads to a contradiction.
10. $\therefore \angle A > \angle D$.	10. Since $\angle A$ is not $= \angle D$, or $< \angle D$.

Note. Additional Exercises 66–68, page 278, can be studied now.

Miscellaneous Exercises

Ex. 217. If two equal oblique segments are drawn to a line from a point in a perpendicular to the line:



(1) they cut off equal distances from the foot of the perpendicular. (Prove $AD = DB$.)

(2) they make equal angles with the perpendicular. (Prove $\angle 1 = \angle 2$.)

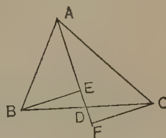
(3) they make equal angles with the given line.

Ex. 218. If lines be drawn through the vertices of a triangle parallel, respectively, to the opposite sides; (*a*) they form a triangle whose sides are bisected by the vertices of the given triangle; (*b*) this triangle is four times as large as the given triangle.

Ex. 219. If two lines are cut by a transversal so that a pair of exterior angles on the same side of the transversal are supplementary, the lines are parallel.

Ex. 220. If perpendiculars be drawn to the sides of an acute angle from a point outside of the angle, they form an angle equal to the given angle.

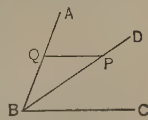
Ex. 221. If D is mid-point of side BC of $\triangle ABC$, and BE and CF are perpendiculars from B and C to AD, extended if necessary, prove $BE = CF$.



Ex. 222. If lines be drawn through the vertices of a quadrilateral parallel to the diagonals of the quadrilateral, they form a parallelogram which is twice as large as the quadrilateral.

Ex. 223. Prove that the segments bisecting the base angles of an isosceles triangle and terminating in the opposite sides are equal.

Ex. 224. If a line be drawn through a point in the bisector of an angle parallel to one side of the angle, the bisector, the parallel, and the other side of the angle form an isosceles triangle.



Ex. 225. If the median to the base of a triangle is perpendicular to the base, the triangle is isosceles.

Ex. 226. If the mid-point of any side of a square is joined to the two vertices of the opposite side, the lines so drawn are equal.

Ex. 227. If AX and CY are altitudes of $\triangle ABC$, then $\angle BCY = \angle BAX$.

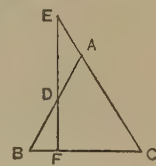
Ex. 228. In isosceles $\triangle ABC$, let $AB = AC$ and $\angle A = 36°$. Prove that the bisector BD of $\angle B$, which meets AC at D, divides $\triangle ABC$ into two isosceles $\triangle$s.

Ex. 229. If each half of the diagonals of a parallelogram is bisected, the segments joining the four mid-points in order form a new parallelogram.

Ex. 230. Construct a parallelogram whose diagonals are 2 in. and 3 in. respectively, if the angle between the diagonals is 45°. Measure the longer and the shorter sides of the parallelogram.

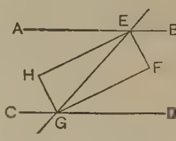
Ex. 231. If the line joining the mid-points of the bases of a trapezoid is perpendicular to the bases, the trapezoid is isosceles.

Ex. 232. If through any point D in one of the equal sides AB of isosceles $\triangle ABC$, DF be drawn perpendicular to base BC, meeting CA extended at E, then $\triangle ADE$ is isosceles.



Suggestion. Compare $\angle E$ with $\angle C$, and $\angle BDF$ with $\angle B$.

Ex. 233. If two parallels are cut by a transversal, the bisectors of the four interior angles form a rectangle.



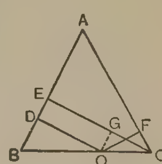
Suggestions. 1. $EFGH$ must be proved a ▱ and one $\angle$ must be proved a right angle.

2. Recall §§ 92, 102, 105.

Ex. 234. Prove that a quadrilateral whose opposite angles are equal is a parallelogram.

Ex. 235. Prove that a quadrilateral is a rectangle if three of its angles are right angles.

Ex. 236. Prove that the sum of the perpendiculars drawn from any point in the base of an isosceles triangle to the equal sides of the triangle is equal to the altitude drawn to one of the equal sides.



Prove $OD + OF = CE$.

Suggestions. 1. Draw $OG \perp CE$.

2. Compare OD and EG; also OF and CG.

Ex. 237. If two triangles have two angles and the bisector of one of these angles equal respectively to two angles and the corresponding bisector of the other, the triangles are congruent.

Suggestion. Recall the note following § 77.

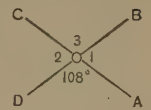
Ex. 238. If two triangles have two sides and the altitude drawn to one of them equal respectively to two sides and the corresponding altitude of the other, the triangles are congruent.

Suggestion. Read the note following § 77.

Note. Additional Exercises 69–96, page 278, can be studied now.

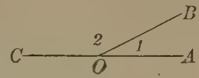
157. Review exercises. Give the answers for the following exercises, with the authorities in full.

Ex. 239. In the adjoining figure, how large is $\angle 2$?



Ex. 240. In the same figure, how large is $\angle 3$?

Ex. 241. In the adjoining figure, if $\angle 1 = 60°$, and $\angle 2 = 120°$, what do you know about OA and OC?



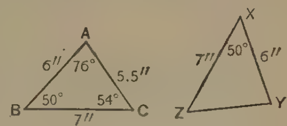
Ex. 242. In the adjoining figure, if $\angle 1 = 30°$, $\angle 4 = 40°$, and $\angle 3 = 50°$, how large is $\angle 2$?

Ex. 243. In the adjoining figure:

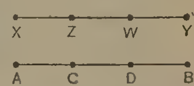
(*a*) What do you know about $\triangle ABC$ and $\triangle XYZ$?

(*b*) What do you know about $\angle Y$?

(*c*) What do you know about $\angle Z$?



Ex. 244. If $XY = AB$; if Z and W divide XY into three equal parts; and if C and D divide AB into three equal parts, what do you know about XZ and DB?



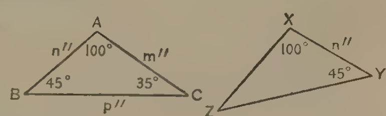
Ex. 245. In the figure of Exercise 244, if $XW = CB$ and $WY = AC$, what do you know about XY and AB?

Ex. 246. In the adjoining figure:

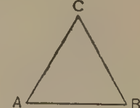
(*a*) What do you know about $\triangle ABC$ and $\triangle XYZ$?

(*b*) What do you know about XZ?

(*c*) What do you know about YZ?



Ex. 247. In the adjoining figure, if $AC = 11''$, if $\angle A = 50°$, and $BC = 11''$, how large is $\angle B$?



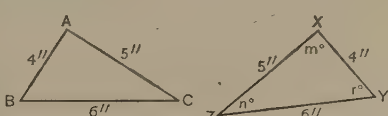
Ex. 248. In the figure adjoining, if $AB = 9''$, if $BC = 9''$, and if $AC = 9''$, how large is $\angle A$?

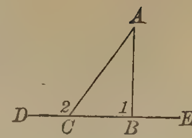
Ex. 249. In the adjoining figure:

(*a*) What do you know about $\triangle ABC$ and $\triangle XYZ$?

(*b*) What do you know about $\angle A$?

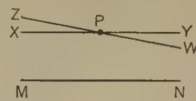
(*c*) What do you know about $\angle B$?



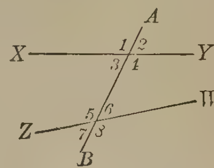


Ex. 250. If $\angle 1$ of the adjoining figure is a right angle, what kind of angle is exterior angle 2?

Ex. 251. If, in the adjoining figure, $\angle 1$ = a rt. $\angle$, and $\angle A = 30°$, how large is $\angle 2$?

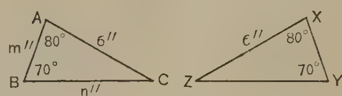


Ex. 252. If $XY \parallel MN$, what do you know about ZW?



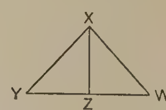
Ex. 253. In the adjoining figure, what would be true about XY and ZW:

(*a*) If $\angle 4 = 100°$ and $\angle 5 = 100°$?
(*b*) If $\angle 2 = 75°$ and $\angle 6 = 75°$?
(*c*) If $\angle 3 = 70°$ and $\angle 5 = 110°$?



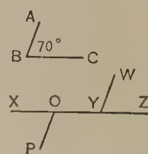
Ex. 254. In the adjoining figure:

(*a*) What do you know about $\triangle ABC$ and $\triangle XYZ$?
(*b*) What do you know about YZ?
(*c*) What do you know about XY?



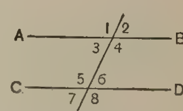
Ex. 255. In the adjoining figure, if $XZ \perp YW$, if $YZ = ZW$, and if $XY = 3''$, how long is XW?

Ex. 256. If in Exercise 255, there is a point P such that $PY = 8''$ and $PW = 8''$, where must P lie?



Ex. 257. In the adjoining figure, if $WY \parallel AB$, if $OP \parallel AB$, and if $XZ \parallel BC$:

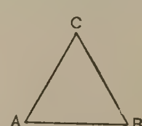
(*a*) How large is $\angle ZYW$?
(*b*) How large is $\angle POZ$?
(*c*) How large is $\angle XOP$?



Ex. 258. In the adjoining figure, if $AB \parallel CD$:

(*a*) How large is $\angle 6$ if $\angle 3 = 80°$?
(*b*) How large is $\angle 8$ if $\angle 4 = 105°$?
(*c*) How large is $\angle 3$ if $\angle 5 = 120°$?

Ex. 259. On a sheet of paper ruled by one set of parallel lines, a line is drawn at right angles to one of these parallels. How does it intersect the other parallels?



Ex. 260. In the adjoining figure, if $\angle A = 40°$, $\angle B = 40°$, and $AC = 5''$, how long is BC?

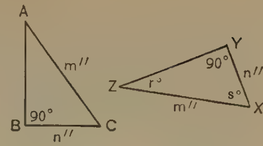
Ex. 261. In the adjoining figure, if $\angle A = 60°$, if $\angle B = 60°$, and $AB = 3''$, how long is AC?

Ex. 262. In the adjoining figure:

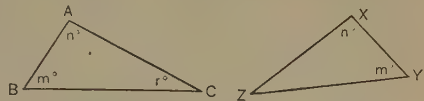
(*a*) What do you know about $\triangle ABC$ and $\triangle XYZ$?

(*b*) What do you know about $\angle C$?

(*c*) What do you know about $\angle A$?

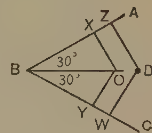


Ex. 263. In the adjoining figure, what do you know about $\angle Z$?



Ex. 264. If $XO \perp AB$; if $OY \perp BC$; and $XO = a''$, how long is OY?

Ex. 265. If $DZ \perp AB$; if $DZ = 7''$; if $DW \perp BC$; if $DW = 7''$; where must D lie?



Ex. 266. In the adjoining figure, if $AB = 3''$; if $AB \parallel CD$; if $CD = 3''$;

(*a*) What do you know about $ABCD$?

(*b*) What do you know about BC?

Ex. 267. In the figure for Exercise 266, if BC and AD each $= 6''$, and if AB and CD each equal $3''$:

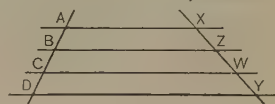
(*a*) What do you know about the figure $ABCD$?

(*b*) What do you know about $\angle A$ and $\angle C$?

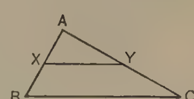
(*c*) What do you know about $\angle A$ and $\angle B$?

Ex. 268. In Exercise 266, if you were to draw AC and BD, what do you know about them?

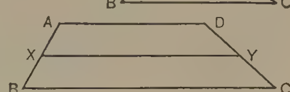
Ex. 269. If $AX \parallel BZ \parallel CW \parallel DY$; if $AB = BC = CD$; and if $XY = 12''$; how long is ZW?



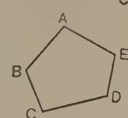
Ex. 270. If $AX = XB = 4''$; if $XY \parallel BC$; and $AY = 5''$; how long is YC?



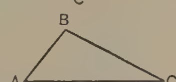
Ex. 271. In the adjoining figure, if $ABCD$ is a trapezoid; if $AX = XB = 3.5''$; if $DY = 4''$; how long is YC?



Ex. 272. In the adjoining figure; if $\angle A = 120°$; if $\angle B = 105°$; if $\angle C = 100°$; if $\angle D = 112°$; how large is $\angle E$?



Ex. 273. If $AB = 4''$, and $BC = 5''$, what do you know about the length of AC?



Ex. 274. If $AB = 4''$, $BC = 5''$, and $\angle A = 75°$, what do you know about $\angle C$?

OPTIONAL TOPICS *

158. Three optional topics follow.

Topic A. **The line joining the mid-points of two sides of a triangle, and associated theorems.**

This important theorem is omitted from the list of theorems recommended by the College Entrance Requirements Board and by the National Committee on Mathematical Requirements. It may therefore be considered optional, but teachers will certainly want their good and excellent pupils to study at least some of this topic. *The first theorem should be studied if Topic B is to be studied.*

Topic B. **Remarkable Points of a Triangle.**

The theorems of this group are not required as authorities in the proofs of fundamental theorems of subsequent Books, and may therefore be considered optional. On the other hand, they do appear in the unstarred list of the College Entrance Requirements Board, and in the list of subsidiary theorems of the National Committee. They may be omitted from a minimum course which is not designed to be college preparatory.

Topic C. **Discussion of Methods of Proof.**

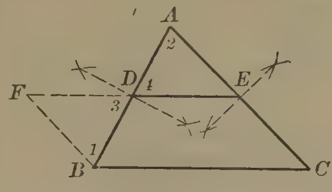
Additional exercise with the indirect method of proof and with analysis as a means of arriving at an original demonstration of a theorem is provided to give increased skill to the better students.

* Note. Attention is called again to page iv of the Preface on which the function of these optional topics is discussed. One method of handling these topics is to allow the capable pupils to go ahead at this point, doing individual work, while the others are given additional time on the minimum course found in pages 1 to 85 inclusive.

Optional Topic A

PROPOSITION XXXV. THEOREM

159. *If a line joins the mid-points of two sides of a triangle, it is parallel to the third side and equal to one half of it.*



Hypothesis. In $\triangle ABC$, $AD = DB$ and $AE = EC$.

Conclusion. $DE \parallel BC$; $DE = \frac{1}{2} BC$.

Plan. Make $FE = 2\,DE$. Try to prove $FE = BC$, and $FE \parallel BC$, by proving $FECB$ is a $\square$

Proof: STATEMENTS	REASONS
1. Extend ED making $DF = ED$. Draw BF.	1. All possible constructions.
2. $\triangle FBD \cong \triangle DAE$.	2. Give full proof.
3. $\therefore \angle 1 = \angle 2$.	3. Why?
4. $\therefore BF \parallel AC$, or $BF \parallel EC$.	4. Why?
5. $BF = AE$.	5. Why?
6. $\therefore BF = EC$.	6. Why?
7. $\therefore BFEC$ is a $\square$.	7. Why?
8. $\therefore FE$ or $DE \parallel BC$.	8. § 124.
9. $FE = BC$.	9. Why?
10. $\therefore DE = \frac{1}{2} BC$.	10. Why?

160. **Fundamental Plan III.** To prove that one segment is double another, either double the shorter and prove the result equal to the longer, or halve the longer and prove the result equal to the shorter. The first of these plans is followed in the proof of Proposition XXXV; the second plan will be used in Proposition XL.

Ex. 275. The lines joining the mid-points of the sides of a triangle divide it into four congruent triangles.

Ex. 276. Prove that the line which joins the mid-points of two sides of a triangle bisects any segment drawn to the third side from the opposite vertex.

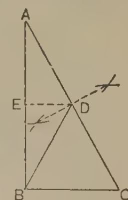
Ex. 277. Prove that the lines drawn from the mid-point of the base of an isosceles triangle to the mid-points of the sides of the triangle form with the half sides a rhombus.

Ex. 278. If D is any point in side AC of $\triangle ABC$ and E, F, G, and H are the mid-points of AD, CD, BC, and AB, respectively, then $EFGH$ is a parallelogram. *Suggestion.* Draw BD.

Ex. 279. If E, F, G, and H are the mid-points of the sides AB, BC, CD, and AD respectively of a quadrilateral $ABCD$, then $EFGH$ is a parallelogram. (Draw AC and use Proposition XXXV.)

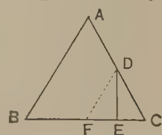
Ex. 280. The lines joining the mid-points of the opposite sides of a quadrilateral bisect each other.

Ex. 281. The mid-point of the hypotenuse of a right triangle is equidistant from the vertices of the triangle. (Let $AE = EB$. Prove $ED \perp AB$. Then complete the proof.)



Ex. 282. If D is the mid-point of side AC of isosceles $\triangle ABC$, and DE is perpendicular to base BC, then EC is $\frac{1}{4}$ BC.

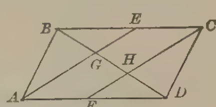
Suggestion. Draw DF parallel to AB.



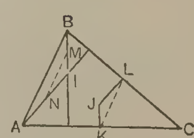
Ex. 283. If the lower base AD of trapezoid $ABCD$ is double the upper base BC, and the diagonals intersect at E, prove that CE is $\frac{1}{2}$ AE and that BE is $\frac{1}{2}$ ED.

Ex. 284. If E and F are the mid-points of BC and AD respectively of parallelogram $ABCD$, prove that AE and CF trisect BD.

Suggestion. Prove $AE \parallel FC$, by proving $AECF$ is a parallelogram. Then prove that AE bisects BH and CF bisects GD.



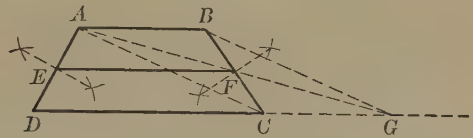
Ex. 285. If I is the intersection point of two altitudes of $\triangle ABC$, and J of the perpendicular bisectors of the corresponding sides, then $BI = 2\,JK$, and $AI = 2\,JL$.



Suggestion. Draw MN connecting the mid-points of BI and AI respectively; also draw KL. Prove $\triangle MNI \cong \triangle JKL$.

PROPOSITION XXXVI. THEOREM

161. *The median of a trapezoid is parallel to the bases and equal to one half their sum.*

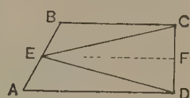


Hypothesis. $ABCD$ is a trapezoid.
E is the mid-point of AD and F of BC.

Conclusion. $EF \parallel AB$ and DC.
$EF = \frac{1}{2}(AB + DC)$.

Plan. Construct a segment equal to $AB + DC$, and prove EF equal to $\frac{1}{2}$ of it and parallel to it.

Proof: STATEMENTS	REASONS
1. Extend DC to G; make $CG = AB$. Draw AC, AG, and BG.	1. § 10.
2. $ABGC$ is a $\square$.	2. § 133. Give the proof.
3. $\therefore AG$ passes through F, and $AF = FG$.	3. § 131.
4. But $AE = ED$.	4. By the hypothesis.
5. $\therefore EF \parallel DG$, or $EF \parallel DC$.	5. § 159.
6. $\therefore EF \parallel AB$.	6. § 90.
7. Also, $EF = \frac{1}{2} DG$.	7. Why?
8. $\therefore EF = \frac{1}{2}(AB + DC)$.	8. $DG = AB + DC$, by Step 1.



Ex. 286. $ABCD$ is a trapezoid whose parallel sides AD and BC are perpendicular to CD. If E is the mid-point of AB, prove $EC = ED$.

Suggestion. Draw EF parallel to AD.

Ex. 287. (Plan for a second proof of § 161.)

In trapezoid $ABCD$, above, having mid-points E and F, draw AC. From E, draw $EXY \parallel DC$, cutting AC at X and BC at Y. Prove EXY coincides with EF and equals $\frac{1}{2}(AB + CD)$.

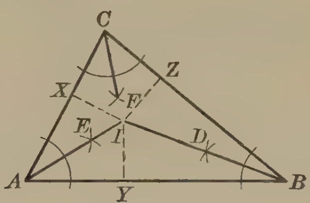
Note. Additional Exercises 97–102, p. 280, can be studied now.

Optional Topic B

162. Three or more lines ordinarily do not pass through a common point. Three or more lines which *do pass* through a common point are called **concurrent lines.**

PROPOSITION XXXVII. THEOREM

163. *The bisectors of the interior angles of a triangle meet at a point which is equidistant from the sides of the triangle.*



Hypothesis. In $\triangle ABC$:

AD bisects $\angle A$; BE bisects $\angle B$; CF bisects $\angle C$.

Conclusion. AD, BE, and CF meet at a point which is equidistant from the sides of $\triangle ABC$.

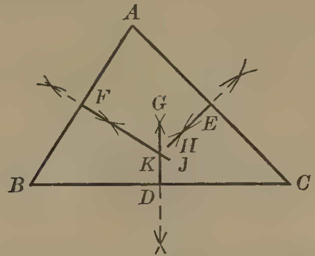
Proof: Statements	Reasons
1. Let AD intersect BE at I.	1. See Note 1, below.
2. Draw $IX \perp AC$; $IY \perp AB$; $IZ \perp BC$.	2. § 82.
3. $IX = IY$.	3. § 117, I.
4. $IZ = IY$.	4. § 117, I.
5. $\therefore IX = IZ$.	5. Ax. 1, § 49.
6. $\therefore I$ lies in CF.	6. § 117, II.
7. $\therefore AD$, BE, and CF meet at I, which is equidistant from AB, AC, and BC.	

Note 1. If AD does not intersect BE, then $AD \parallel BE$, and then $\angle DAB + \angle EBA = 1$ st. $\angle$. (§ 97.) But $\angle DAB + \angle EBA < 1$ st. $\angle$.

Note 2. The point of intersection of the bisectors of the interior angles of a triangle is called the **in-center** of the triangle.

PROPOSITION XXXVIII. THEOREM

164. *The perpendicular-bisectors of the sides of a triangle meet at a point which is equidistant from the vertices of the triangle.*



Hypothesis. In $\triangle ABC$, FJ, DG, and EH are the perpendicular-bisectors of AB, BC, and AC, respectively.

Conclusion. FJ, DG, and EH meet at a point which is equidistant from A, B, and C.

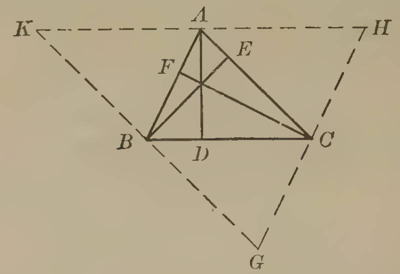
Proof: STATEMENTS	REASONS
1. Let FJ intersect DG at K.	1. See Note 1.
2. Draw AK, BK, and CK.	2. § 5, *a*.
3. Since K is on FJ, $AK = BK$.	3. § 116, I.
4. Since K is on DG, $BK = CK$.	4. § 116, I.
5. $\therefore AK = CK$.	5. Ax. 1, § 49.
6. $\therefore K$ lies on EH.	6. § 116, II.
7. $\therefore FJ$, DG, and EH meet at K, which is equidistant from A, B, and C.	

Note 1. 1. If FJ does not intersect GD, then $FJ \parallel GD$.
2. $\therefore AB$, which is $\perp$ to FJ, is also $\perp$ to GD.
3. But $BD \perp GD$.
4. $\therefore$ either $AB \parallel BD$, or AB coincides with BD.
5. But this is impossible, since AB and BD intersect.

Note 2. The point of intersection of the perpendicular-bisectors of the sides of a triangle is called the **circum-center** of the triangle, for a circle can be drawn with it as center which will pass through the vertices of the triangle.

PROPOSITION XXXIX. THEOREM

165. *The altitudes of a triangle meet at a point.*



Hypothesis. In $\triangle ABC$, AD, BE, and CF are the altitudes from A, B, and C respectively.

Conclusion. AD, BE, and CF meet at a point.

Plan. Form a second triangle, which has AD, BE, and CF as the perpendicular bisectors of its sides.

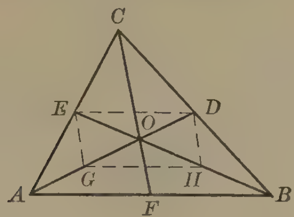
Proof: STATEMENTS	REASONS
1. Through A, draw $KH \parallel BC$; through B, draw $KG \parallel AC$; through C, draw $HG \parallel AB$.	1. § 89.
2. Since $AD \perp BC$, $AD \perp HK$.	2. Why?
3. $KA = BC$ and $AH = BC$.	3. Why?
4. $\therefore KA = AH$.	4. Why?
5. $\therefore AD$ is the $\perp$-bis. of KH.	5. § 76.
6. Also BE is the $\perp$-bis. of KG and FC is the $\perp$-bis. of HG.	6. Similar reasoning.
7. $\therefore AD$, BE, and CF are concurrent.	7. § 164.

Note. The point of intersection of the altitudes of a triangle is called the ortho-center of the triangle.

Ex. 288. Draw a large triangle. With compasses and ruler construct the lines which determine the in-center, the circum-center, and the ortho-center. Mark these centers by the letters I, K, and O.

PROPOSITION XL. THEOREM

166. *The medians of a triangle meet at a point which lies two thirds the distance from each vertex to the mid-point of the opposite side.*



Hypothesis. AD, BE, and CF are the medians of $\triangle ABC$.

Conclusion. AD, BE, and CF meet at a point which lies two thirds the distance from each vertex to the mid-point of the opposite side.

Proof:

STATEMENTS	REASONS
1. Let AD and BE meet at point O.	1. Obvious.
2. Let G and H be the mid-points of AO and BO respectively. Draw ED, GH, EG, DH.	2. Possible.
3. Then in $\triangle AOB, GH = \frac{1}{2} AB$ and $GH \parallel AB$.	3. Why?
4. Similarly, $ED = \frac{1}{2} AB$ and $ED \parallel AB$.	
5. $\therefore EDHG$ is a ▱.	5. Why?
6. $\therefore GD$ and EH bisect each other.	6. Why?
7. $\therefore OD = OG = AG$, and $EO = OH = HB$.	

8. Hence AD and BE meet at a point which lies two thirds the distance from A to D and from B to E.

9. In like manner, AD and CF meet at a point which lies two thirds the distance from A to D. This is point O.

10. Hence the three medians meet at point O, which is two thirds the distance from each vertex to the mid-point of the opposite side.

Note. Point O is called the center of gravity of the triangle.

NOTE. Additional Exercises 103–105, p. 281, can be studied now.

Optional Topic C

Methods of Proof

167. The **indirect method** of proof, described in § 93, page 48, is also called the *reductio ad absurdum* proof. Review that section, and then solve the following exercises.

Ex. 289. If two lines are cut by a transversal, and a pair of alternate interior angles are unequal, the lines are not parallel.

Ex. 290. If two lines are cut by a transversal and the sum of the interior angles on the same side of the transversal is not equal to two right angles, the lines are not parallel.

Ex. 291. If a point is unequally distant from the ends of a segment, it is not in the perpendicular-bisector of the segment.

Ex. 292. If a point is not equidistant from the sides of an angle, it is not in the bisector of the angle.

Ex. 293. If two unequal oblique segments be drawn to a line from a point on a perpendicular to the line, they cut off on the line unequal distances from the foot of the perpendicular, the greater cutting off the greater distance. (See Ex. 215.)

Ex. 294. If two lines are not parallel, and are cut by a transversal, alternate-interior angles are not equal.

Ex. 295. If two adjacent angles are not supplementary, their exterior sides are not in a straight line.

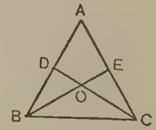
Ex. 296. If a point is not on the perpendicular-bisector of a segment, it is not equally distant from the ends of the segment.

Ex. 297. If three parallels cut off two unequal segments on one transversal, then they cut off two unequal segments on any other transversal.

168. Analysis as a means of arriving at a proof.

Ex. 298. Hyp. BE and CD are medians of $\triangle ABC$.
BE and CD are equal.

Con. $\triangle ABC$ is isosceles, with $AB = AC$.

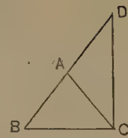


Analysis. 1. If $AB = AC$, then $BD = CE$, *and conversely*.

2. This suggests trying to prove $\triangle BDO \cong \triangle OEC$.

(Give the proof now in the regular form, omitting this analysis, stating the plan, and giving the statements and the reasons as usual. As a help, what do you know about BO as a result of § 166?)

Ex. 299. If AC be drawn from the vertex of the right angle to the hypotenuse of right $\triangle BCD$ so as to make $\angle ACD = \angle D$, it bisects the hypotenuse.



Hyp. $\angle BCD =$ rt. $\angle$. **Con.** $BA = AD$.
$\angle ACD = \angle D$.

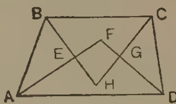
Analysis. 1. Since $\angle ACD = \angle D$, $AD = AC$, *and conversely.*

2. If $AB = AD$, then $AB = AC$, *and conversely.*

3. If $AB = AC$, then $\angle B = \angle ACB$, *and conversely.*

4. If now you can prove $\angle B = \angle ACB$, then you can work back through the Steps 3, 2, and 1 *since each was true conversely.* Give this proof in the usual form, omitting the analysis. Use § 108 and § 38.

Ex. 300. If the bisectors of the interior angles of a trapezoid do not meet at a point, they form a quadrilateral, two of whose angles are right angles.



Hyp. $ABCD$ is a trapezoid, with $BC \parallel AD$.
AF, BH, CH, and DF bisect their angles.

Con. $\angle FEH$ and $\angle FGH$ are rt. $\angle$s.

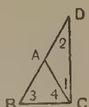
Analysis. 1. If $\angle FEH =$ rt. $\angle$, then $\angle BEA =$ rt. $\angle$, *and conversely.*

2. If $\angle BEA =$ rt. $\angle$, then $\angle ABE + \angle BAE =$ rt. $\angle$, *and conversely.*

3. If $\angle ABE + \angle BAE =$ rt. $\angle$, then $\angle ABC + \angle BAD =$ st. $\angle$, *and conversely.*

4. If now you can prove that $\angle ABC + \angle BAD =$ st. $\angle$, then, since each statement was true conversely, you can work back to the desired conclusion. Give this proof in the customary manner, omitting the analysis, and starting with Step 4.

Ex. 301. $AB = AC$ in $\triangle ABC$. BA is extended to D, so that $AD = AB$. Prove that CD is $\perp$ BC.



Analysis. 1. If $CD \perp BC$, then $\angle 1 + \angle 4 =$ rt. $\angle$, *and conversely.*

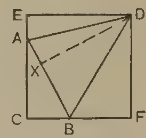
2. Since $\angle 1 + \angle 2 + \angle 3 + \angle 4 = 2$ rt. $\angle$s, if $\angle 1 + \angle 4 =$ rt. $\angle$, then $\angle 1 + \angle 4 = \angle 2 + \angle 3$, *and conversely.*

3. Since $\angle 3 = \angle 4$, if $\angle 1 + \angle 4 = \angle 2 + \angle 3$, then $\angle 1 = \angle 2$, *and conversely.*

4. If now you can prove that $\angle 1 = \angle 2$, then you can work back to the conclusion, since each statement was true conversely. Give the proof, omitting the analysis.

169. The method of analysis taught in Exercises 298–301 should be combined with the suggestions made in § 114 and § 115.

Ex. 302. If AD and BD are the bisectors of the exterior angles at the ends of the hypotenuse AB of right $\triangle ABC$, and DE and DF are perpendiculars respectively to CA and CB extended, then $CEDF$ is a square.



Hyp. $\angle ACB$ = rt. $\angle$; AD bisects $\angle EAB$;
BD bisects $\angle ABF$; $DE \perp CA$; $DF \perp CB$.

Con. $CEDF$ is a square.

Analysis. 1. What must I prove about $CEDF$?

Answer. Prove $CEDF$ is a $\square$, having one rt. $\angle$, and two adj. sides equal.

2. If $DF \parallel CE$, then $DF \perp CF$, and conversely. Similarly prove $DE \parallel CF$.

3. If $DE = DF$, then DF should $= DX$, since $DE = DX$, the $\perp$ to AB. This suggests trying to prove DE and DF both equal to DX.

Ex. 303. Prove that two isosceles triangles are congruent if the vertex angle and its bisector in one are equal to the vertex angle and its bisector in the other.

Ex. 304. If a line be drawn parallel to the base of an isosceles triangle, cutting the two sides, it, the base, and the parts of the sides cut off form an isosceles trapezoid.

Ex. 305. If $ABCD$ is a parallelogram and if BF on $AB = DE$ on DC, then EF and AC bisect each other.

Ex. 306. The bisectors of the opposite angles of a parallelogram are parallel.

Ex. 307. In $\square$ $CDEF$, DR, the bisector of $\angle CDE$, meets CF at R. Prove $\triangle CDR$ is isosceles.

Ex. 308. Prove that the diagonals of an oblique angled parallelogram are not equal.

Ex. 309. If $XYZW$ is a parallelogram; if YA, the bisector of $\angle Y$ meets XW at A, and XB, the bisector of $\angle X$, meets YZ at B, then YA and XB bisect each other.

BOOK II

THE CIRCLE

170. Review the definitions, postulate, and theorem given in §§ 15–19 on page 8.

The symbol for "circle" is $\odot$.

The circle whose center is O is indicated thus: $\odot\, O$.

171. Since a circle is a closed plane line (§ 15), it incloses a portion of the plane. This portion is called the **interior of the circle.**

Ex. 1. Draw a circle of radius 1 in. Where will a point lie:

(*a*) if its distance from the center is $\frac{3}{4}$ in.?

(*b*) if its distance is 1.5 in.?

Ex. 2. Draw a circle of diameter 3 in. Cut the circular piece from the paper. Prove, by folding it on a diameter, that any diameter bisects the circle and the surface inside it.

Ex. 3. Draw two circles which intersect. From one of the points of intersection draw the radius of each circle.

(*a*) How does the distance between their centers compare with the sum of their radii?

(*b*) How does it compare with the difference of their radii?

172. The following facts are now evident:

(*a*) *All radii and all diameters of the same circle or of equal circles are equal.*

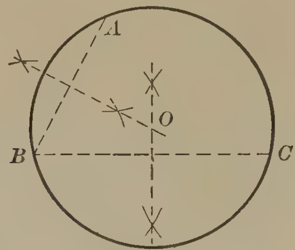
(*b*) *A point is within, on, or outside a circle according as its distance from the center is less than, equal to, or greater than the radius.*

(*c*) *A diameter of a circle bisects the circle and the surface inclosed by it; also, if a line bisects a circle, it is a diameter.*

Ex. 4. Give the definition of: (*a*) circle; (*b*) equal circles; (*c*) radius; (*d*) diameter; (*e*) chord; (*f*) arc; (*g*) minor arc.

* PROPOSITION I. PROBLEM

173. *Construct a circle which will pass through three points which are not in a straight line.*



Given points A, B, and C which are not in a straight line.

Required to construct a circle which will pass through A, B, and C.

Construction. 1. Draw AB and BC.

2. Construct the $\perp$ bisectors of AB and BC, meeting at O.

Statement. A circle drawn with O as center and OA as radius will pass through A, B, and C.

Proof. $OA = OB = OC$. § 116, I.

Ex. 5. What would happen if A, B, and C were in a straight line?

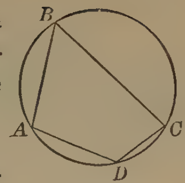
174. ***Cor. 1.** *Only one circle can be drawn through three points which are not in a straight line.*

For, the $\perp$-bisectors of AB and BC can meet at only one point.

175. Cor. 2. *A circle cannot be drawn through three points which do lie in a straight line.*

For, if A, B, and C are in a straight line, the perpendicular-bisectors of AB and BC would be parallel. Hence there is not a point which is equidistant from A, B, and C.

176. Cor. 3. *A straight line cannot intersect a circle in more than two points.*



177. A **polygon** is said to be **inscribed in a circle** when its vertices lie on the circle; as $ABCD$. The **circle** is said to be **circumscribed about the polygon.**

178. One half of a circle is called a **semicircle.**

A quarter of a circle is called a **quadrant.**

Circles having the same center are called **concentric circles.**

SOME CONCENTRIC CIRCLE DESIGNS

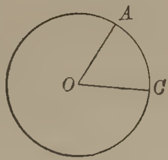
Ex. 6. Construct a triangle having sides 3 in., 4 in., and 2.5 in. long. Circumscribe a circle about the triangle. Measure its radius.

Ex. 7. (*a*) Construct a circle which will pass through two given points.

(*b*) Construct two other circles through the two given points.

Ex. 8. How many circles can be constructed through two points?

Ex. 9. Construct full size the adjoining design for a four-inch square tile.



179. A **central angle** is an angle whose vertex is at the center and whose sides are radii of the circle; as $\angle COA$.

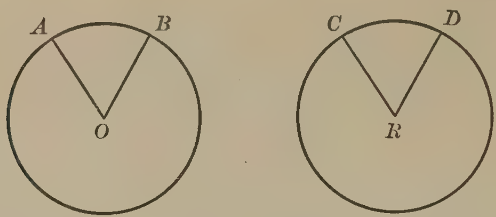
$\angle COA$ is said to *intercept* $\overset{\frown}{CA}$.

$\overset{\frown}{CA}$ is said to be *intercepted by* $\angle COA$.

"Intercept" means "to take between."

PROPOSITION II. THEOREM

180. *In the same circle or in equal circles, if central angles are equal, they intercept equal arcs.*



Hypothesis. $\odot O = \odot R$; $\angle BOA = \angle DRC$.

Conclusion. $\overset{\frown}{BA} = \overset{\frown}{DC}$.

Plan. Use the superposition method of proof.

Proof:

Statements	Reasons
1. Place $\odot O$ on $\odot R$, so that $\angle BOA$ coincides with its equal, $\angle DRC$.	1. § 60 § 22.
2. $\therefore$ $\odot O$ will coincide with $\odot R$.	2. § 18.
3. Point A will fall on point C.	3. $OA = RC$, by hyp.
4. Point B will fall on point D.	4. Why?
5. $\therefore$ $\overset{\frown}{BA} = \overset{\frown}{DC}$.	5. Since they coincide.

Ex. 10. Divide a circle into four equal arcs.

Suggestion. What kind of central $\angle$s must be constructed? Draw two perpendicular diameters.

Ex. 11. Divide a circle into six equal arcs.

Suggestion. How large must the central $\angle$s be? Draw a radius. Recall Ex. 123, and Ex. 130, Book I, pp. 54.

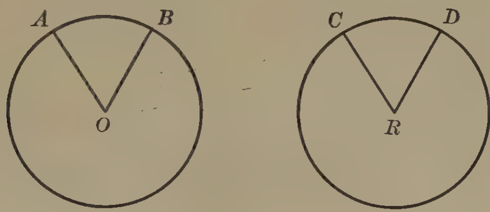
Ex. 12. Tell how you can divide a circle into five equal arcs, using your protractor.

Ex. 13. By means of compasses, ruler, and protractor construct a five-pointed star in a circle of 2 in. radius, to be used as a pattern for a star on a sailor collar.

Ex. 14. Divide a circle into eight equal arcs.

PROPOSITION III. THEOREM

181. *In the same circle or in equal circles, if arcs are equal, the central angles which intercept them are equal.*



Hypothesis. $\odot O = \odot R$; $\overset{\frown}{BA} = \overset{\frown}{DC}$.

Conclusion. $\angle BOA = \angle DRC$.

Plan. Use the superposition method of proof.

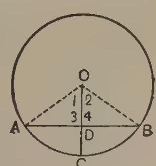
Proof: STATEMENTS	REASONS
1. Place $\odot O$ so that it coincides with $\odot R$, point O on R, and A on C.	1. Equal ⓢ can be made to coincide.
2. $\therefore$ point B falls on point D.	2. $\overset{\frown}{AB} = \overset{\frown}{CD}$, by hyp.
3. $\therefore$ OA falls on RC, and OB on RD.	3. Why?
4. $\therefore$ $\angle BOA = \angle DRC$.	4. Why?

182. It may be proved by superposition that: *in the same circle or in equal circles:*

(*a*) *The greater of two unequal central angles intercepts the greater arc;*

(*b*) *The greater of two unequal arcs is intercepted by the greater central angle.*

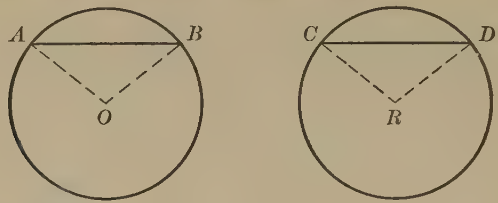
183. *A* chord *AB* is said to subtend arc *AB*. Arc *AB* is said to be subtended by chord *AB*.



Ex. 15. If a radius bisects an arc, it is perpendicular to and bisects the chord which subtends the arc.

PROPOSITION IV. THEOREM

184. *In the same circle or in equal circles, if chords are equal, they subtend* equal arcs.*



Hypothesis. $\odot O = \odot R$; $AB = CD$.

Conclusion. $\widehat{AB} = \widehat{CD}$.

Plan. 1. Draw AO, OB, RC, RD.

2. Prove $\angle O = \angle R$, and apply § 180.

[Proof to be given by the pupil.]

Ex. **16.** Construct an equilateral triangle. Circumscribe a circle about the triangle. (§ 173.) Prove that the vertices of the triangle divide the circle into three equal arcs.

PROPOSITION V. THEOREM

185. *In the same circle or in equal circles, if arcs are equal, the chords which subtend them are equal.*

Hypothesis. $\odot O = \odot R$; $\widehat{AB} = \widehat{CD}$. (Fig. § 184.)

Conclusion. $AB = CD$.

Plan. Compare $\angle O$ and $\angle R$, using § 181. Prove $\triangle AOB \cong \triangle CRD$.

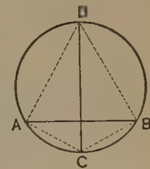
[Proof to be given by the pupil.]

Ex. **17.** If a radius be drawn from the center of a circle to the midpoint of an arc, it is perpendicular to and bisects the chord of that arc.

* Note. "Subtend" is derived from Latin words meaning "to stretch under."

186. Fundamental Plan IV. To prove arcs equal, prove their central angles are equal, or their chords are equal; to prove chords equal, prove their arcs are equal; to prove central angles are equal, prove their arcs are equal.

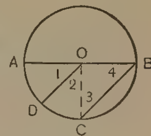
Ex. 18. If two diameters of a circle be drawn, they divide the circle into two pairs of equal arcs.



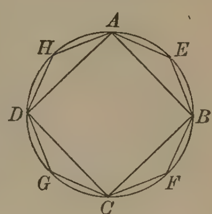
Ex. 19. Prove that the straight line connecting the mid-points of the two arcs which are subtended by one chord is perpendicular to and bisects the chord.

Ex. 20. (*a*) Construct an isosceles triangle having $AB = AC = 2$ in., and $BC = 1.5$ in., and circumscribe about it a circle. Measure the radius of the circle.

(*b*) Prove that arc AB of the circle equals arc AC.



Ex. 21. In circle O, chord BC is drawn from one end of diameter AB. Radius OD is parallel to BC. Prove that arc AD = arc DC.



Ex. 22. In the adjoining figure, $ABCD$ is a square. E, F, G, and H are the mid-points of arcs AB, BC, CD, and AD respectively. Prove:
(*a*) that $AEBFCGDH$ is an equilateral polygon.
(*b*) that $\triangle$s AEB, BCF, CDG, and DAH are congruent isosceles triangles.

Ex. 23. In Exercise 22, let O be the center of the circle. How does $\angle BOA$ compare with $\angle BOE$?

Ex. 24. From the mid-point of an arc, perpendiculars are drawn to the radii which are drawn to the ends of the arc. Prove that these perpendiculars are equal.

Ex. 25. If A, B, C, and D are four points in order on a circle, and if $\overparen{AB} = \overparen{CD}$, prove that chord AC = chord BD.

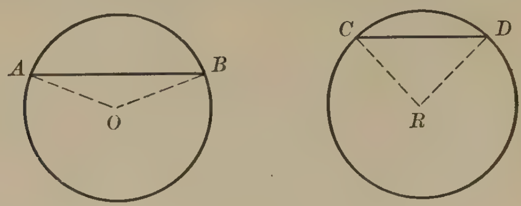
Ex. 26. Assume that the circle O is divided into six equal arcs. Prove that the chords of these arcs and the radii drawn to the points of division form six congruent equilateral triangles.

Ex. 27. In Exercise 26, how many degrees are there in the angle formed by two radii? How many in the angle formed by two consecutive chords?

Note. Additional Exercises 1–9, page 282, can be done now.

PROPOSITION VI. THEOREM

187. *In the same circle or in equal circles, if two minor arcs are unequal, then their chords are unequal, the greater arc being subtended by the greater chord.*



Hypothesis. $\odot O = \odot R$; $\overset{\frown}{AB} > \overset{\frown}{CD}$.

Conclusion. $AB > CD$.

Plan. Prove $\angle BOA > \angle DRC$, and use § 155, p. 79.

Proof:

STATEMENTS	REASONS
1. Draw AO, BO, CR, and DR. In $\triangle AOB$ and $\triangle CRD$:	1. § 5, *a*.
2. $AO = CR$, and $BO = DR$.	2. Why?
3. $\overset{\frown}{AB} > \overset{\frown}{CD}$.	3. Why?
4. $\therefore \angle O > \angle R$.	4. § 182, *b*.
5. $\therefore AB > CD$.	5. § 155.

PROPOSITION VII. THEOREM

188. *In the same circle or in equal circles, if two chords are unequal, then they subtend unequal minor arcs, the greater chord subtending the greater arc.*

Hypothesis. $\odot O = \odot R$; $AB > CD$. (Fig. § 187.)

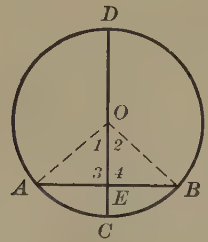
Conclusion. $\overset{\frown}{AB} > \overset{\frown}{CD}$.

Plan. Prove $\angle O > \angle R$ (§ 156) and apply § 182, *a*.

Ex. 28. Construct $\triangle ABC$, having $AB = 2$ in., $BC = 3$ in., and $AC = 2.5$ in. Circumscribe a circle with center O about $\triangle ABC$. Prove: (*a*) that $\overset{\frown}{AB} < \overset{\frown}{AC} < \overset{\frown}{BC}$; (*b*) that $\angle BOC > \angle COA > \angle AOB$.

* PROPOSITION VIII. THEOREM

189. *If a diameter is perpendicular to a chord, it bisects the chord and its subtended arcs.*



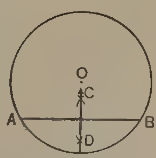
Hypothesis. In $\odot O$, diameter $CD \perp AB$ at E.

Conclusion. $AE = EB$; $\overset{\frown}{AC} = \overset{\frown}{CB}$; and $\overset{\frown}{AD} = \overset{\frown}{DB}$.

Plan. Prove AE and EB, and $\angle 1$ and $\angle 2$ corres. parts of $\cong \triangle$s.

Proof: Statements	Reasons
1. Draw OA and OB.	1. § 5, *a*.
2. $\triangle AOE \cong \triangle OEB$.	2. Give the full proof.
3. $\therefore AE = EB$.	3. Why?
4. Also $\angle 1 = \angle 2$.	4. Why?
5. $\therefore \overset{\frown}{AC} = \overset{\frown}{CB}$.	5. Why?
6. $\overset{\frown}{CAD} = \overset{\frown}{CBD}$.	6. § 172, *c*.
7. $\overset{\frown}{AD} = \overset{\frown}{DB}$.	7. Why?

190. Cor. *The perpendicular-bisector of a chord passes through the center of the circle, and bisects the arcs subtended by the chord.*



Plan. Compare AO and OB. Then use § 116, II.

Ex. 29. If a radius of a circle bisects a chord, it is perpendicular to the chord, and bisects the subtended arcs.

(Additional exercises appear on page 106.)

Ex. 30. Given an arc of a circle. Bisect it.

Ex. 31. Given an arc of a circle, of unknown center and unknown radius. Find the center and the radius of the circle.

Suggestion. Draw two chords, and use § 190.

Ex. 32. If a line is drawn from the center of a circle perpendicular to a chord which is not a diameter of the circle, it bisects the chord; and, if it is extended, it also bisects the arc subtended by the chord.

Ex. 33. If a line is drawn from the center of a circle to the mid-point of a chord, it is perpendicular to the chord; and, if it is extended, it bisects the arcs subtended by the chord.

Ex. 34. A radius drawn to the mid-point of an arc is perpendicular to the chord which subtends the arc.

Ex. 35. If a straight line joins the mid-point of a chord and the mid-point of the arc subtended by the chord, it is perpendicular to the chord; and if extended, it passes through the center of the circle.

Ex. 36. If a straight line is drawn cutting two concentric circles in points A, B, C, and D respectively, then $AB = CD$.

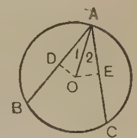
Suggestion. Draw a $\perp$ to $ABCD$ from the center of the circles.

Ex. 37. Prove that the perpendicular-bisectors of the sides of any inscribed polygon all pass through the center of the circle.

Suggestion. Recall § 190.

Ex. 38. Inside a circle place a point P. Through P construct a chord which will be bisected by point P.

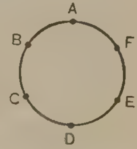
Ex. 39. If a diameter of a circle bisects each of two chords, the chords are parallel.



Ex. 40. If two chords of a circle, drawn from the same point of the circle, make equal angles with the radius drawn to the point, the chords are equal.

Suggestion. From the center, draw perpendiculars to the two chords.

Ex. 41. If two equal chords are drawn from a point of a circle, the radius drawn to the point bisects the angle between the two chords.

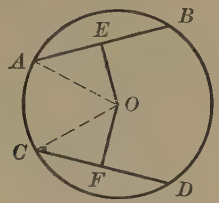


Ex. 42. Suppose that A, B, C, D, E, and F divide circle O into six equal arcs. Draw chords AB, BC, CD, DE, EF, and FA. Prove that polygon $ABCDEF$ is equilateral.

Ex. 43. If a radius is perpendicular to a chord, it bisects the angle formed by joining the outer end of the radius to the ends of the chord.

* PROPOSITION IX. THEOREM

191. *In the same circle or in equal circles, if chords are equal, they are equidistant from the center.*



Hypothesis. In $\odot ABC$:

$AB = CD$; $OE \perp AB$; $OF \perp CD$.

Conclusion. $OE = OF$.

Plan. Prove OE and OF corres. parts of cong. $\triangle$s.

Proof:

STATEMENTS	REASONS
1. Draw OA and OC.	1. Why possible?
2. $\therefore AE = \frac{1}{2} AB$; $CF = \frac{1}{2} CD$.	2. Why?
3. But $AB = CD$.	3. Why?
4. $\therefore AE = CF$.	4. Why?
5. $\therefore \triangle AOE \cong \triangle OCF$.	5. Give the proof.
6. $\therefore OE = OF$.	6. Why?

Ex. 44. If two equal chords intersect inside the circle, the radius drawn to the point of intersection bisects the angle between the chords.

Suggestion. Draw perpendiculars to the chords from the center.

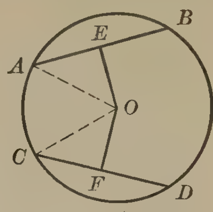
Ex. 45. Construct an equilateral triangle whose sides are 3 in. long. Circumscribe a circle about it, thus locating the center of the circle. Construct perpendiculars to the three sides from the center. Prove these perpendiculars are equal.

Ex. 46. In the figure for Ex. 42, page 106, draw perpendiculars to AB, BC, CD, etc. What must be true about these perpendiculars?

Ex. 47. Draw any circle and divide it into four equal arcs (Ex. 10). Draw the chords of these arcs, and prove them equidistant from the center of the circle.

* PROPOSITION X. THEOREM

192. *In the same circle or in equal circles, if chords are equidistant from the center, they are equal.*



Hypothesis. In $\odot ABC$:

$OE \perp AB$; $OF \perp CD$; $OE = OF$.

Conclusion. $AB = CD$.

Plan. Prove AE and CF corres. parts of cong. $\triangle$s.

Proof:

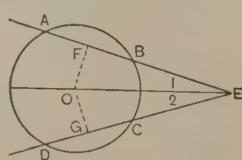
STATEMENTS	REASONS
1. Draw OA and OC.	1. Why possible.
2. $\therefore \triangle OAE \cong \triangle OCF$.	2. Give the proof.
3. $\therefore AE = CF$.	3. Why?
4. $AB = 2\,AE$; $CD = 2\,CF$.	4. Why?
5. $\therefore AB = CD$.	5. Why?

Ex. 48. If two chords, intersecting within the circle, make equal angles with the radius passing through their point of intersection, they are equal.

Suggestion. Draw $\perp$s to the chords from the center.

Ex. 49. If a straight line be drawn parallel to the line connecting the centers of two equal circles, and intersecting the circles, then the chords formed in the two circles are equal.

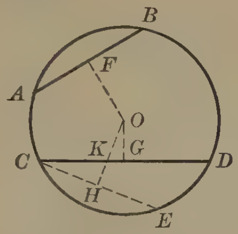
193. A straight line which intersects a circle in two points is called a **secant** of the circle.



Ex. 50. If ABE and DCE are two secants of a circle which make equal angles with the line connecting E with the center of the circle, then chord AB = chord DC. (Use § 192.)

PROPOSITION XI. THEOREM

194. *In the same circle or in equal circles, the less of two unequal chords is at the greater distance from the center of the circle.*



Hypothesis. In $\odot O$:

$AB < CD$; $OF \perp AB$; $OG \perp CD$.

Conclusion. $OF > OG$.

Proof: STATEMENTS	REASONS
1. $AB < CD$.	1. Why?
2. $\therefore \overset{\frown}{AB} < \overset{\frown}{CD}$.	2. Why?
3. Let $\overset{\frown}{CE} = \overset{\frown}{AB}$. Draw CE.	3. $\overset{\frown}{AB} < \overset{\frown}{CD}$.
4. $CE = AB$.	4. Why?
5. Draw $OH \perp CE$, cutting CD at K.	5. § 82. OH cuts CD, since O and H are on opposite sides of CD.
6. $\therefore OH = OF$.	6. Why?
7. But $OH > OK$.	7. Ax. 8, § 49.
8. $\therefore OF > OK$.	8. Why?
9. Also $OK > OG$.	9. § 154.
10. $\therefore OF > OG$.	10. Ax. 11, § 148.

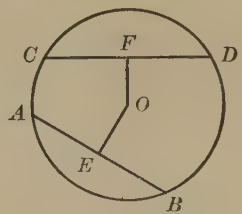
Ex. 51. If two equal chords of a circle intersect inside the circle, the segments of one are equal respectively to the segments of the other.

Ex. 52. An equilateral triangle and a square are inscribed in a circle. Prove that the sides of the triangle are nearer the center than the sides of the square.

Ex. 53. AB is a diameter of a circle and XY is an intersecting diameter of a smaller concentric circle. Prove $AXBY$ is a $\square$.

PROPOSITION XII. THEOREM

195. *In the same circle or in equal circles, if two chords are unequally distant from the center, the more remote is the smaller.*



Hypothesis. In $\odot O$:

$OE \perp AB$; $OF \perp CD$; $OE > OF$.

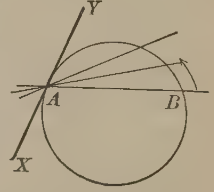
Conclusion. $AB < CD$.

Plan. Use the indirect method of proof.

Proof:

STATEMENTS	REASONS
1. Suppose AB not $< CD$.	1. AB $<$, or not $< CD$.
2. If $AB = CD$, then $OE = OF$.	2. Why?
3. But $OE > OF$.	3. Why?
4. $\therefore AB$ does not $= CD$.	4. See § 93.
5. If $AB > CD$, then $OE < OF$.	5. § 194.
6. But $OE > OF$.	6. Why?
7. $\therefore AB$ is not $> CD$.	7. See § 93.
8. $\therefore AB < CD$.	8. See Steps 4 and 7.

196. Tangent line. Let secant AB turn about the point A in the direction indicated by the arrow. Point B moves closer to point A. When B finally coincides with A, the line assumes the position XY.



XY is called a tangent to the circle.

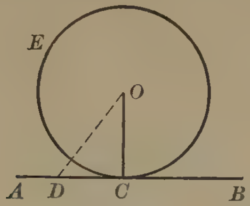
A tangent to a circle is said to touch the circle at only one point.

The circle is said to be **tangent to the line.**

Point A is the **point of tangency** or **point of contact.**

PROPOSITION XIII. THEOREM

197. *A straight line perpendicular to a radius at its outer extremity is tangent to the circle.*



Hypothesis. OC is a radius of $\odot O$; $AB \perp OC$ at C.

Conclusion. AB is tangent to $\odot O$.

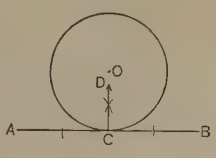
Plan. Prove all points of AB except C are outside $\odot O$.

Proof: STATEMENTS	REASONS
1. Draw OD from O to any point D of AB except point C.	1. Why possible?
2. $\therefore OD$ is not $\perp AB$.	2. § 87. $OC \perp AB$, by Hyp.
3. $\therefore OD > OC$.	3. Why?
4. $\therefore D$ lies outside $\odot O$.	4. § 172, *b*.
5. $\therefore AB$ is tangent $\odot O$.	5. § 196.

198. Cor. 1. *A tangent to a circle is perpendicular to the radius drawn to the point of contact.*

Since all points in AB except C lie outside the circle, OC is the shortest segment to AB from O. Hence $OC \perp AB$.

199. Cor. 2. *A line perpendicular to a tangent at its point of contact passes through the center of the circle.*

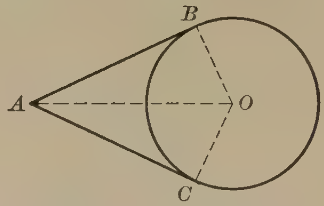


By Cor. 1, the radius OC is $\perp$ to AB. Hence OC and CD must coincide. (Why?) $\therefore CD$ passes through the center of the circle.

200. Cor. 3. *A line from the center of the circle perpendicular to a tangent passes through the point of contact.*

PROPOSITION XIV. THEOREM

201. *The tangents to a circle from an outside point are equal.*



Hypothesis. AB and AC are tangents to $\odot O$ from A.

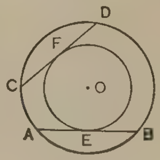
Conclusion. $AB = AC$.

[Plan and proof to be given by the pupil.]

202. Cor. *The two tangents to a circle from an outside point make equal angles with the line from that point to the center of the circle.*

Note. This theorem does not appear in Euclid at all. It appears first as a definite theorem in writings of Hero, although it was apparently used by Archimedes. The proof of Prop. XIV is attributed to a mathematician Fink, about 1583.

Ex. 54. Prove that the tangents to a circle at the extremities of a diameter are parallel.

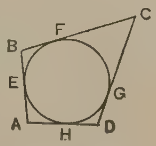


Ex. 55. If two circles are concentric, any two chords of the greater which are tangents of the smaller are equal.

Ex. 56. Prove that all tangents drawn from the larger of two concentric circles to the smaller are equal.

Suggestion. Prove $CF = AE$.

Ex. 57. Prove that the line joining the center of a circle to the point of intersection of two tangents: (*a*) bisects the angle formed by the radii drawn to the points of contact; (*b*) bisects and is perpendicular to the chord joining the points of contact.



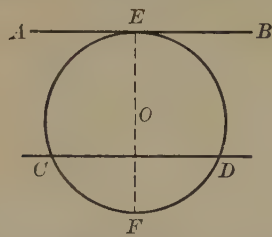
Ex. 58. Prove that the sum of two opposite sides of a circumscribed quadrilateral is equal to the sum of the other two opposite sides.

Note. Additional Exercises 10–15, page 282, can be done now.

PROPOSITION XV. THEOREM

203. *Parallel lines intercept equal arcs on a circle.*

Case I. *When one line is a tangent and one a secant:*



Hypothesis. AB is tangent to $\odot\, O$ at E;
secant $CD \parallel AB$.

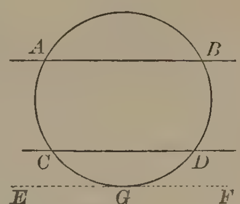
Conclusion. $\overset{\frown}{CE} = \overset{\frown}{DE}$.

Plan. Prove diameter $EOF \perp CD$, and bisects $\overset{\frown}{CED}$.

Proof:

Statements	Reasons
1. Draw diameter EOF.	1. Why possible?
2. $\therefore EF \perp AB$.	2. Why?
3. $\therefore EF \perp CD$.	3. Why?
4. $\therefore \overset{\frown}{CE} = \overset{\frown}{ED}$.	4. § 189.

Case II. *When both lines are secants:*

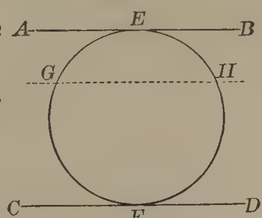


Hyp. AB and CD are $\parallel$ secants of the $\odot$.

Con. $\overset{\frown}{AC} = \overset{\frown}{BD}$.

Plan. Assume EGF tangent to the $\odot$ at G, and parallel to CD.

Case III. *When both lines are tangents:*



Hyp. AEB and CFD are $\parallel$ tangents to the $\odot$ at E and F.

Con. $\overset{\frown}{EGF} = \overset{\frown}{EHF}$.

[Proof to be given by the pupil.]

MEASUREMENT OF ANGLES AND ARCS

204. To measure a given magnitude, two steps are necessary.

(*a*) Select a quantity of the same kind to be used as the **unit of measure.**

(*b*) Determine the number of times the given magnitude contains the unit of measure. This number is called the **numerical measure** of the quantity in terms of the unit employed.

Thus, to measure a segment AB, we may select the unit 1 foot. If it is contained in AB six times, then the numerical measure of AB is 6, for the unit 1 foot. Or we may select the unit 1 yard. Then it will be contained in AB only twice, and the numerical measure of AB is 2, for the unit 1 yard.

205. Two magnitudes of the same kind are **commensurable** when each contains the same unit of measure, called a **common measure**, an *integral* number of times.

Thus, two segments whose lengths are $2\frac{1}{2}$ in. and $3\frac{1}{4}$ in. respectively are commensurable, for the common measure $\frac{1}{4}$ in. is contained in the first segment 10 times and in the second 13 times.

Two magnitudes of the same kind are said to be **incommensurable** when no unit of measure can be found which is contained an *integral* number of times in each.

The diagonal and the side of a square are incommensurable.

206. The **ratio** of two magnitudes of the same kind is the quotient of their numerical measures in terms of a common measure.

Thus, the segments of lengths $2\frac{1}{2}$ in. and $3\frac{1}{4}$ in. have the ratio $\frac{10}{13}$.

Ex. 59. What is the measure of a yard in terms of the unit: (*a*) 1 ft.? (*b*) 1 in.? (*c*) $\frac{1}{2}$ in.?

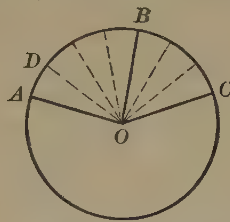
Ex. 60. What is the ratio of 2 yd. to $1\frac{1}{2}$ ft.?

Ex. 61. (*a*) How does the ratio of one quart to one gallon compare with the ratio of one peck to one bushel?

(*b*) How does the ratio of one side of a square to the perimeter of the square compare with the ratio of one right angle to 360°?

PROPOSITION XVI. THEOREM

207. *In the same circle or in equal circles, two central angles have the same ratio as their intercepted arcs.*



CASE I. *When the angles are commensurable.*

Hypothesis. In $\odot O$, $\angle BOA$ and $\angle COB$ are commensurable.

Conclusion. $\dfrac{\angle COB}{\angle BOA} = \dfrac{\overset{\frown}{CB}}{\overset{\frown}{BA}}$.

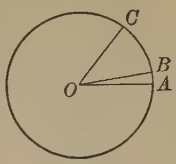
Plan. Measure the $\measuredangle$ and $\frown$s; find the ratios.

Proof: STATEMENTS	REASONS
1. Let $\angle DOA$ be the common measure of $\angle COB$ and $\angle BOA$.	1. They are assumed to be commensurable.
2. Let $\angle COB = 3\angle DOA$; $\angle BOA = 4\angle DOA$.	2. $\angle DOA$ is a common measure.
3. $\therefore \dfrac{\angle COB}{\angle BOA} = \dfrac{3}{4}$.	3. § 206.
4. The radii in Step 2 divide $\overset{\frown}{CB}$ into 3 and $\overset{\frown}{BA}$ into 4 equal arcs.	4. Why?
5. $\therefore \dfrac{\overset{\frown}{CB}}{\overset{\frown}{BA}} = \dfrac{3}{4}$.	5. § 206.
6. $\therefore \dfrac{\angle COB}{\angle BOA} = \dfrac{\overset{\frown}{CB}}{\overset{\frown}{BA}}$.	6. Ax. 1, § 49.

CASE II. *When the angles are incommensurable:*

The theorem is true also in this case. (Proof on p. 266.)

208. Measuring angles and arcs. In § 28, the unit for measuring angles is given as 1 degree, $\frac{1}{90}$ of a right angle. This will be called for the present one *angular-degree.* Let $\angle AOB$ represent 1°. 60 angular-seconds equal one angular-minute; and 60 angular-minutes equal one angular-degree.



Let a circle be drawn around point O as center, and the radii which divide the total angle into 360 equal central angles be imagined. These angles are angular-degrees. They will intercept 360 equal arcs on the circle. Let $\overset{\frown}{AB}$ represent one of these arcs. It is the unit for measuring arcs *on this circle and on any equal circle.* It will be called one *arc-degree.*

Evidently on a circle with longer radius, the arc corresponding to $\overset{\frown}{AB}$ will be longer.

209. A central angle has the same measure as its intercepted arc, *when angular-degrees and arc-degrees are used as the respective units of measure.*

Let $\angle AOB$ above represent 1 angular-degree and $\angle AOC$ any other central angle.

Then $$\frac{\angle AOC}{\angle AOB} = \frac{\overset{\frown}{AC}}{\overset{\frown}{AB}}. \qquad \S\ 207$$

But $\dfrac{\angle AOC}{\angle AOB}$ is the numerical measure $\angle AOC$, and $\dfrac{\overset{\frown}{AC}}{\overset{\frown}{AB}}$ is the numerical measure of $\overset{\frown}{AC}$, by the definition.

Hence the measure of $\angle AOC$ equals the measure of $\overset{\frown}{AC}$.

Thus, if $\angle AOC = 57$ angular-degrees, then $\overset{\frown}{AC} = 57$ arc-degrees.

210. *From now on, it will be understood that angles are measured in terms of angular-degrees, and arcs in terms of arc-degrees.* Also, the following statement of the theorem of § 209 will be employed for convenience:

A central angle is measured by (i. m. b.) its intercepted arc.

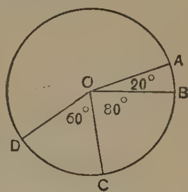
Ex. 62. What is an arc-degree? An angular-degree?

Ex. 63. Are all angular-degrees of the same size?

(*a*) Are all arc-degrees of the same size on the same or on equal circles? (*b*) On unequal circles?

Ex. 64. If $ABCD$ is an inscribed square and O is the center of the circle, how many degrees are there in $\widehat{AB}$? In $\angle AOB$?

Ex. 65. $\triangle ABC$ is an equilateral triangle inscribed in a circle with center O. How many degrees are there in $\widehat{AB}$? In $\angle AOB$?



Ex. 66. In the adjoining figure, compare $\widehat{AB}$ and $\widehat{BC}$. Also compare $\widehat{AB}$ and $\widehat{DC}$; also $\widehat{BC}$ and $\widehat{DC}$.

Ex. 67. A right central angle is what part of a straight angle? What part therefore is its intercepted arc of a semicircle?

Ex. 68. A 60° angle is what part of the perigon? What part therefore is its intercepted arc of the whole circle?

Ex. 69. If a circle is divided into 3 equal parts, what part of the perigon is the central angle which intercepts one of the parts? How many degrees are there in the central angle?

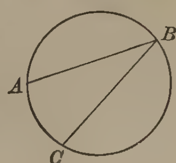
Ex. 70. If AB, any chord of a circle O, is extended to a point C so that BC equals the radius of the circle, and CO is drawn, cutting the circle at Z and E respectively, then $\widehat{AE} = 3\ \widehat{BZ}$.

Suggestion. 1. Draw OA and OB. 2. Prove $\angle AOE = 3\ \angle BOZ$, using § 109, and § 68.

Ex. 71. In the figure for Ex. 42, page 106, how many degrees are there in central angle AOB? In $\angle AOE$?

Ex. 72. If radii be drawn to the points which divide a circle into five equal arcs, how large is each of the central angles?

211. An angle is said to be an **inscribed angle** when its vertex is on the circle and its sides are chords of the circle; as $\angle ABC$.



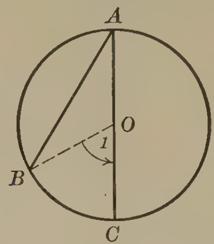
$\angle ABC$ intercepts the $\widehat{AC}$; $\widehat{AC}$ is intercepted by the $\angle ABC$.

$\angle ABC$ is said to be inscribed in the circle or may be said to be inscribed in the arc ABC.

* PROPOSITION XVII. THEOREM

212. *An inscribed angle is measured by one half its intercepted arc.*

CASE I. *When the center lies on one side of the angle:*



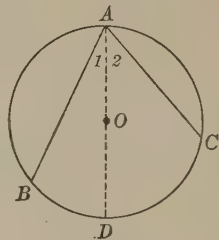
Hypothesis. Center O lies on side AC of $\angle BAC$.

Conclusion. $\angle BAC$ is measured by $\frac{1}{2}\ \overset{\frown}{BC}$.

Proof:

	STATEMENTS	REASONS
1.	Draw BO forming $\angle BOC$.	1. Why possible.
2.	$\angle 1 = \angle B + \angle A$.	2. Why?
3.	$\therefore\ \angle 1 = 2\ \angle A$, or $\angle A = \frac{1}{2}\ \angle 1$.	3. Give the proof.
4.	$\angle 1$ is measured by $\overset{\frown}{BC}$.	4. § 210.
5.	$\therefore\ \angle A$ is measured by $\frac{1}{2}\ \overset{\frown}{BC}$.	5. Ax. 2, § 49.

CASE II. *When the center lies inside the angle:*



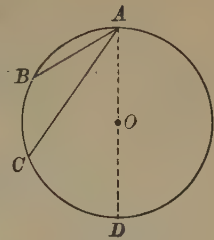
Hypothesis. Center O lies inside inscribed $\angle BAC$.

Conclusion. $\angle BAC$ is measured by $\frac{1}{2}\ \overset{\frown}{BC}$.

Plan. Draw AOD. Use Case I.

Proof:

	STATEMENTS	REASONS
1.	Draw diameter AOD.	1. Why possible?
2.	$\angle 1$ i. m. b. $\frac{1}{2}\ \overset{\frown}{BD}$, and $\angle 2$ i. m. b. $\frac{1}{2}\ \overset{\frown}{DC}$.	2. By Case I.
3.	$\therefore (\angle 1 + \angle 2)$ i. m. b. $\frac{1}{2}(\overset{\frown}{BD} + \overset{\frown}{DC})$.	3. Ax. 3, § 49.
4.	$\therefore\ \angle BAC$ i. m. b. $\frac{1}{2}\ \overset{\frown}{BC}$.	4. Ax. 7, § 49.



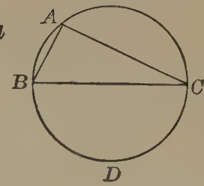
CASE III. *When the center lies outside the angle:*

Hyp. Center O lies outside inscribed $\angle BAC$.

Con. $\angle BAC$ is measured by $\frac{1}{2}\overset{\frown}{BC}$.

Plan. Draw AD. Measure $\angle BAD$ and $\angle CAD$. Subtract $\angle CAD$ from $\angle BAD$.

[Give the proof.]



213. Cor. 1. *An angle inscribed in a semicircle is a right angle.*

Hyp. BC is a diameter.
$\angle BAC$ is an inscribed $\angle$.

Con. $\angle BAC$ is a rt. $\angle$.

214. Cor. 2. *Inscribed angles which intercept the same arc are equal.*

Plan. Draw two or more inscribed angles which intercept the same arc. Prove them equal.

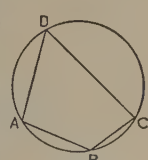
Ex. 73. How many degrees are there in $\angle BAC$ in Case I: (*a*) if $\overset{\frown}{BC} = 60°$? (*b*) if $\angle BOC = 45°$? (*c*) if $\overset{\frown}{BA} = 110°$?

Ex. 74. How many degrees are there in $\angle BAC$ in Case II: (*a*) if $\overset{\frown}{BDC} = 110°$? (*b*) if $\overset{\frown}{AB} = 100°$ and $\overset{\frown}{AC} = 110°$?

Ex. 75. How many degrees are there in $\angle BAC$ in Case III: (*a*) if $\overset{\frown}{BC} = 50°$? (*b*) if $\overset{\frown}{BA} = 15°$ and $\overset{\frown}{AC} = 110°$? (*c*) if $\overset{\frown}{BA} = 15°$ and $\angle CAD = 40°$?

Ex. 76. Three consecutive sides of an inscribed quadrilateral subtend arcs of 82°, 90°, and 60° respectively. Find each angle of the quadrilateral.

Ex. 77. If chords AB and CD intersect at E within the circle, prove that $\triangle AEC$ and $\triangle BDE$ are mutually equiangular.



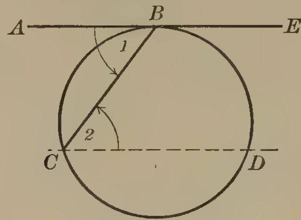
Ex. 78. If chords AB and CD extended meet outside the circle at point E, prove $\triangle ADE$ and $\triangle BCE$ are mutually equiangular.

Ex. 79. Prove that the opposite angles of an inscribed quadrilateral are supplementary.
(By what is $\angle B$ measured? $\angle D$? $\therefore \angle B + \angle D$?)

Note. Additional Exercises 16 to 38, page 283, can be studied now.

PROPOSITION XVIII. THEOREM

215. *The angle formed by a tangent and a chord drawn to the point of contact is measured by one half its intercepted arc.*



Hypothesis. AE is tangent to $\odot\, CBD$ at B; BC is a chord.

Conclusion. $\angle ABC$ is measured by $\frac{1}{2}\,\overset{\frown}{BC}$.

Plan. Find the measure of an $\angle$ which equals $\angle ABC$.

Proof:

STATEMENTS	REASONS
1. Draw $CD \parallel AE$, through C.	1. Possible by § 98.
2. $\angle ABC = \angle BCD$.	2. Why?
3. $\angle BCD$ i. m. b. $\frac{1}{2}\,\overset{\frown}{BD}$.	3. Why?
4. But $\overset{\frown}{BC} = \overset{\frown}{BD}$.	4. § 203.
5. $\therefore\ \angle ABC$ i. m. b. $\frac{1}{2}\,\overset{\frown}{BC}$.	5. Ax. 2, § 49.

Ex. 80. How many degrees are there in $\angle ABC$ and $\angle EBC$ above: (*a*) if $\overset{\frown}{BC} = 100°$? (*b*) if $\overset{\frown}{CD} = 75°$? (*c*) if $\angle BCD = 43°$?

Ex. 81. If tangents are drawn to a circle at the extremities of a chord, they make equal angles with the chord.

Ex. 82. If two tangents drawn from a point to a circle form an angle of 60°, then each of the tangents equals the chord joining the points of contact. (Prove the triangle is equilateral.)

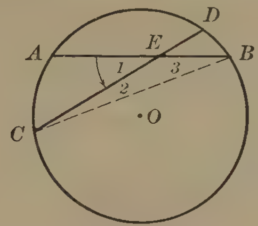
Ex. 83. If a tangent be drawn to a circle at the extremity of a chord, the line joining the mid-point of the intercepted arc to the point of contact bisects the angle formed by the tangent and the chord.

Ex. 84. Prove that a tangent to a circle at the mid-point of an arc is parallel to the chord of the arc.

Note. Additional Exercises 39–43, page 285, can be done now.

PROPOSITION XIX. THEOREM

216. *The angle formed by two chords intersecting within a circle is measured by one half the sum of the arcs intercepted by it and its vertical angle.*



Hypothesis. Chords AB and CD intersect at E within $\odot O$.

Conclusion. $\angle AEC$ is measured by $\frac{1}{2}(\widehat{AC} + \widehat{BD})$.

Plan. Measure two angles whose sum equals $\angle AEC$.

Proof:

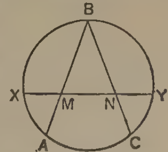
STATEMENTS	REASONS
1. Draw CB, forming $\triangle CEB$.	1. Why possible?
2. Then $\angle 1 = \angle 2 + \angle 3$.	2. Why?
3. $\angle 2$ i. m. b. $\frac{1}{2}\widehat{DB}$; $\angle 3$ i. m. b. $\frac{1}{2}\widehat{AC}$.	3. Why?
4. $\therefore$ $\angle 1$ i. m. b. $\frac{1}{2}(\widehat{AC} + \widehat{DB})$.	4. Ax. 2, § 49.

Ex. 85. How large is $\angle AEC$ if $\widehat{AC} = 70°$ and $\widehat{DB} = 50°$?

Ex. 86. How large is $\widehat{DB}$ if $\widehat{AC} = 74°$ and $\angle AEC = 40°$?

Ex. 87. How large is $\angle AED$ if $\widehat{AC} = 35°$ and $\widehat{DB} = 25°$?

Ex. 88. If two chords intersect at right angles within a circle, the sum of one pair of opposite intercepted arcs is equal to a semicircle.



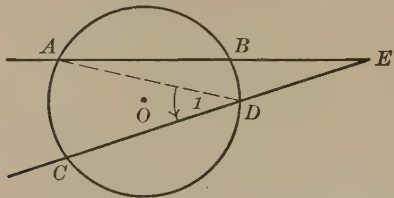
Ex. 89. If $\widehat{XA} = 40°$, $\widehat{YC} = 40°$, $\widehat{AC} = 60°$, and $\widehat{BX} = \widehat{BY}$, prove $\triangle BMN$ is isosceles, and find the size of each of its angles.

Ex. 90. If $\widehat{AX} = \widehat{CY}$ and $\widehat{AB} = \widehat{CB}$, prove $\triangle MNB$ is an isosceles triangle.

Note. Additional Exercises 44 to 46, page 285, can be studied now.

PROPOSITION XX. THEOREM

217. *The angle formed by two secants intersecting outside the circle is measured by one half the difference between its intercepted arcs.*



Hypothesis. Secants AB and CD intersect at E outside $\odot O$.

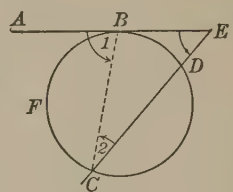
Conclusion. $\angle E$ is measured by $\frac{1}{2}(\widehat{AC} - \widehat{DB})$.

Plan. Express $\angle E$ as the difference of two $\angle$s which can be measured.

Proof:

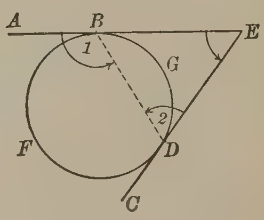
STATEMENTS	REASONS
1. Draw AD, forming $\triangle ADE$.	1. Why possible?
2. $\therefore \angle A + \angle E = \angle 1$.	2. Why?
3. $\therefore \angle E = \angle 1 - \angle A$.	3. Why?

(Complete the proof.)



218. Cor. 1. *The angle formed by a secant and a tangent is measured by one half the difference between its intercepted arcs.*

Prove $\angle E$ is measured by $\frac{1}{2}(\widehat{BC} - \widehat{DB})$.



219. Cor. 2. *The angle formed by two tangents is measured by one half the difference between its intercepted arcs.*

Prove $\angle E$ is measured by $\frac{1}{2}(\widehat{BFD} - \widehat{BGD})$.

Ex. 91. In § 217, if $\widehat{AC} = 100°$ and $\widehat{BD} = 40°$, find $\angle E$.

Ex. 92. In § 217, if $\widehat{AC} = 90°$ and $\angle E = 42°$, find $\widehat{BD}$.

Ex. 93. In § 217, if $\widehat{AC} = 120°$ and $\angle A = 20°$, find $\angle E$.

Ex. 94. In § 218, if $\angle E = 60°$ and $\widehat{BD} = 55°$, find $\widehat{BFC}$.

Ex. 95. In § 219, if $\widehat{BFD} = 300°$, how large is $\angle E$?

Ex. 96. In § 218, if $\widehat{BFC} = 210°$ and $\widehat{CD} = 110°$, find $\angle E$.

Ex. 97. Prove that the straight line joining the points of contact of two parallel tangents is a diameter of the circle.

Ex. 98. If AB and AC are two chords making equal angles with the tangent at point A, prove $AB = AC$.

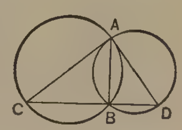
Ex. 99. A square $ABCD$ is inscribed in a circle. A tangent is drawn to the circle at point A. How large is the angle formed by the tangent and the side AB?

Ex. 100. The bisector of the angle formed by a tangent and the chord drawn to the point of contact bisects the intercepted arc.

Ex. 101. $\triangle ABC$ is inscribed in a circle. D and E bisect the arcs AB and AC respectively. DE cuts AB at F and AC at G. Prove that $AF = AG$.

Ex. 102. Prove that two chords which are perpendicular to a third chord, which is not a diameter, are equal.

Ex. 103. Prove that an inscribed trapezoid is isosceles.



Ex. 104. If AB is the common chord of two intersecting circles, and AC and AD are diameters drawn from A, prove that CBD is a straight line.

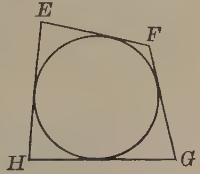
Ex. 105. Prove that the chords joining the ends of two intersecting diameters form a rectangle.

Ex. 106. If a quadrilateral is inscribed in a circle, the exterior angle formed at any vertex by extending one of the sides equals the interior angle at the opposite vertex.

Note. Additional Exercises 47 to 59, page 285, can be studied now.

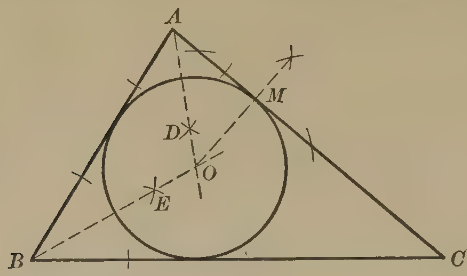
220. A circle is said to be **inscribed in a polygon** when it is tangent to each side of the polygon.

The polygon is said to be **circumscribed about the circle**; as $EFGH$.



PROPOSITION XXI. PROBLEM

221. *Inscribe a circle in a given triangle.*



Given $\triangle ABC$.

Required to inscribe a $\odot$ in $\triangle ABC$.

Construction. 1. Construct the bisectors BE and AD of $\angle B$ and $\angle A$ respectively, meeting at point O.

2. Construct $OM \perp AC$.

3. With O as center and OM as radius, draw a $\odot$.

Statement. This circle will be tangent to AB, BC, and AC.

Proof: STATEMENTS	REASONS
1. O is equidistant from AB, BC, and AC.	1. § 117, I.
2. The distance from O to each side $= OM$.	2. $OM \perp AC$.
3. $\therefore$ AB, BC, and AC are tangent to O.	3. § 197.

Note 1. O is called the **in-center** of the triangle since it is the center of the inscribed circle.

Note 2. A circle can be constructed which is tangent to the sides AB and AC prolonged and to BC. It is called an **escribed circle** and its center is called an **ex-center** of the triangle.

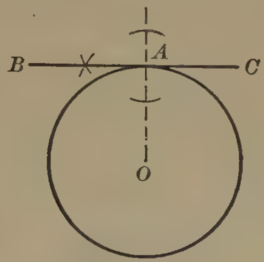
Ex. 107. (*a*) Construct $\triangle ABC$ having $AB = 2$ in., $BC = 3$ in., and $AC = 4$ in.

(*b*) Construct its inscribed circle. Measure its radius.

(*c*) Construct the three escribed circles. Measure their radii.

PROPOSITION XXII. PROBLEM

222. I. *Construct a tangent to a circle at a point on the circle.*

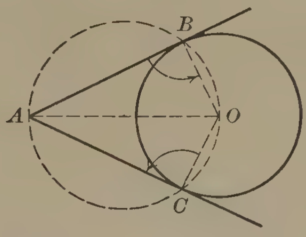


Given ⊙ O and point A on it.

Required to construct a tangent to ⊙ O at A.

[Construction, statement, etc., to be given by the pupil.]

II. *Construct a tangent to a circle from a point outside the circle.*



Given O and point A outside ⊙ O.

Required to construct a tangent to ⊙ O from point A.

Construction. 1. Draw AO.

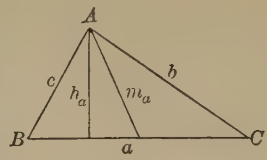
2. Construct a ⊙ on AO as diameter intersecting ⊙ O at B and C.

3. Draw AB and AC.

Statement. AB and AC are both tangent to ⊙ O.

Suggestion. Draw OB and OC. Prove $\angle B$ and $\angle C$ are rt. $\angle$s.

CONSTRUCTION OF TRIANGLES



223. In a $\triangle ABC$, the sides opposite angles A, B, and C are marked by the small letters a, b, and c respectively.

The letter h denotes an altitude. h_a (read h—sub—a) denotes the altitude to side a. Similarly, there are the altitudes h_b and h_c.

The letter m denotes a median. The medians to sides a, b, and c are denoted by m_a, m_b, and m_c respectively.

The letter t is used to denote the length of the bisector of an angle between the vertex and the opposite side. The bisectors of angles A, B, and C are denoted by t_A, t_B, and t_C.

224. A triangle is determined when three independent parts are known.

General Suggestions

1. *Draw freehand a triangle which represents the desired figure, marking with heavy lines the parts which correspond to the given parts. Use this figure as a guide in constructing the desired triangle. It is not the desired triangle, and the parts marked are not necessarily equal in size to the given parts.*
2. *Make the construction, using the given parts.*
3. *Prove that the resulting triangle has all the given parts, and is the kind of triangle specified.*
4. *When the given parts are not of specified size, discuss the construction, determining whether there are conditions under which it may be impossible to construct a triangle having the given parts.*

Note. Review Prop. IV, Book I, page 34.

Ex. 108. Construct $\triangle ABC$ having $AB = 2$ in., $AC = 3$ in., and $\angle A = 40°$. (Use your protractor to draw $\angle A$.) Measure BC.

Ex. 109. Construct $\triangle ABC$ having given sides b and c, and $\angle A$. (Take any short segment for b, and one for c; and any angle for $\angle A$. Follow the suggestions given above.)

Ex. 110. Construct $\triangle ABC$ having $\angle A = 50°$, $\angle B = 70°$, and $AB = 3$ in. When through, measure AC and BC.

Ex. 111. Construct a triangle having given two of its angles and the included side.

Discussion. Consider whether the triangle can be constructed always: (*a*) when both angles are acute; (*b*) when one is acute and one is obtuse.

Ex. 112. (*a*) Construct $\triangle ABC$, having $AB = 3$ in., $\angle C = 50°$, and $\angle A = 70°$. Measure BC and AC.

(*b*) Repeat the exercise when AB is changed to 2 in.

(*c*) What happens to BC and to AC, when AB becomes smaller, if $\angle A$ and $\angle C$ remain unchanged?

Ex. 113. Construct a triangle having a given side, the opposite angle, and another angle.

Ex. 114. (*a*) Construct right triangle ABC, having hypotenuse $AB = 2$ in., and leg $AC = 1.5$ in. Measure BC.

(*b*) Repeat the exercise, changing AB to 2.5 in.

(*c*) What happens to BC if AC remains unchanged and AB becomes larger.

Ex. 115. Construct a right triangle having a given hypotenuse and a given leg.

Ex. 116. Construct right triangle ABC, having hypotenuse $AB = 3$ in., and $\angle A = 40°$. Measure legs AC and BC.

(*b*) Repeat the exercise when $\angle A = 60°$.

(*c*) What happens to AC and to BC as $\angle A$ becomes larger, if AB remains unchanged?

Ex. 117. Construct isosceles triangle ABC, having base $AC = 2$ in., and the altitude to $AC = 2$ in. Measure AB and BC.

(*b*) Repeat the exercise when the base remains unchanged and the altitude becomes 3 in.

(*c*) What happens to AB if the base remains unchanged, and the altitude becomes larger?

Ex. 118. (*a*) Construct the isosceles triangle which has a given vertex angle, and a given altitude from that vertex.

(*b*) Without drawing additional figures, what will happen to the base, as the vertex angle decreases, if the altitude remains unchanged?

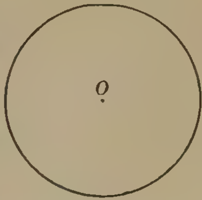
Ex. 119. Construct an equilateral triangle having a given altitude.

Note. For further discussion of construction of figures see pages 143 to 150.

LOCI

225. Illustrative problem. Where are all points $\frac{1}{2}$ in. from O?

Evidently the *place* of points $\frac{1}{2}$ in. from O is the circle with center O and radius $\frac{1}{2}$ in.



Instead of using the word "place" it is customary to use the word *locus* — a Latin word meaning place. So the preceding sentence becomes

The locus of points $\frac{1}{2}$ in. from O is the circle with center O and radius $\frac{1}{2}$ in.

It is evident that:

(*a*) Every point "$\frac{1}{2}$ in. from O" is on the circle.

(*b*) Every point on the circle is "$\frac{1}{2}$ in. from O."

"$\frac{1}{2}$ in. from O" is the condition which the points satisfy.

Ex. 120. Draw the locus of points which are 2 in. from a given point.

Ex. 121. Draw the locus of points outside a circle with 1 in. radius and $\frac{1}{4}$ in. from the circle.

Ex. 122. Draw any line of indefinite length.

(*a*) Locate freehand three points above the line which are 1 in. from the line.

(*b*) Draw the line which contains all points which are 1 in. from the line and lie above the line.

(*c*) Are there any other points which are 1 in. from the line?

(*d*) Draw the line showing where they are to be found.

Ex. 123. Where are, that is, *what is the locus of*, points on this page which are $\frac{1}{4}$ in. from the left-hand edge of the page?

Ex. 124. Draw two parallel lines.

(*a*) Locate freehand three points which are equidistant from the two parallels.

(*b*) Draw the locus of points which are equidistant from the parallels.

Ex. 125. (*a*) Draw a line AB and locate on it a point C. Construct three circles, all tangent to AB at C.

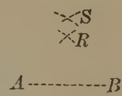
(*b*) What is the locus of the center of a $\odot$ which will be tangent to AB at C?

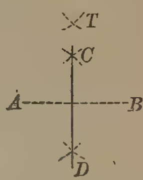
226. Def. If a single geometrical condition is given, the **locus of points** satisfying that condition is the line or group of lines such that:

(*a*) Every point in the line (or lines) satisfies the condition.

(*b*) Every point which satisfies the condition lies in the line or group of lines.

227. *Problem. *Determine the locus of points equidistant from two given points.*





Solution. (*a*) 1. R is located so that $RA = RB$. Similarly, S and T are located.

2. Their position suggests that the locus of such points is the $\perp$-bisector of AB.

(*b*) 1. Assume that CD, the $\perp$-bisector of AB, is the locus of points equidistant from A and B.

2. Is every point on CD equidistant from A and B?

Answer. Yes, by § 116, I.

3. Is every point equidistant from A and B in line CD?

Answer. Yes, by § 116, II.

(*c*) $\therefore$ *The locus of points equidistant from two given points is the perpendicular-bisector of the segment joining the points.*

Note. If preparing for a College Entrance Board Examination, give only part (*c*) and the full proofs of § 116, I and II.

Ex. 126. Locate two points X and Y which are 2 in. apart. Construct the locus of points which are equidistant from X and Y

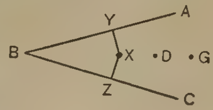
Ex. 127. In a given line, find all points which are equidistant from two points X and Y which are not in the line.

Ex. 128. On a given circle, find all points which are equidistant from the ends of a chord of the circle.

Ex. 129. Find in two parallels all points which are equidistant from X and Y, which are not on the parallels.

Ex. 130. Draw two intersecting straight lines, and locate any points X and Y not on these lines. Find in the lines all points which are equidistant from X and Y.

228. Problem. *Determine the locus of points within an angle which are equidistant from the sides of the angle.*



Solution. (*a*) 1. X is located so that ⊥s XY and XZ are (apparently) equal. Similarly D and G are located.

2. The position of X, D, and G suggest that the locus of such points is the bisector of $\angle CBA$.

(*b*) 1. *Assume* that bisector BR of $\angle CBA$ is the locus of points equidistant from BC and BA.

2. Is every point on BR equidistant from CB and AB?

Answer. Yes, by § 117, I.

3. Is every point which is equidistant from BC and BA on BR?

Answer. Yes, by § 117, II.

(*c*) ∴ *the locus of points within an angle, equidistant from the sides of the angle, is the bisector of the angle.*

Note. If preparing for a College Entrance Board Examination, give only part (*c*) and the full proofs of § 117, I and II.

Ex. 131. Draw two intersecting lines. Construct the locus of points inside each of the four angles which are equidistant from the sides of the angles.

Ex. 132. Draw $\triangle ABC$. In AC, find a point which is equidistant from AB and BC.

Ex. 133. Draw two parallel lines, and an acute angle whose sides intersect them. Find all points inside the angle which are equidistant from the sides of the angle, and also on one of the parallels.

Ex. 134. Draw a circle, and from an exterior point, draw two secants of the circle. Find all points on the circle which are inside the angle formed by the secants, and which are equidistant from the secants.

Ex. 135. (*a*) Locate two points X and Y, and construct the locus of points which are equidistant from X and Y.

(*b*) Draw $\angle ABC$, and construct the locus of points inside $\angle ABC$, which are equidistant from AB and BC.

(*c*) Extend the loci (*a*) and (*b*). If they intersect, call the point P. What do you know about P?

229. Review of the minimum course of Book II is given by Exercises 136 to 161 inclusive. For a hasty review, require only the answers to the exercises; for a more searching review, have each answer proved by the quotation in full of the proper authority.

Ex. 136. Where does a point B lie if its distance from the center of a circle of radius 3 in. is: (*a*) 2 in.? (*b*) 3 in.? (*c*) 4 in.?

Ex. 137. Which of the following statements must be true and which must be false:

(*a*) Circle O goes through X, Y, and Z when Y lies on XZ.

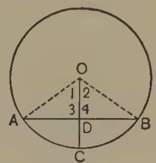
(*b*) Straight line m cuts circle O at points R, S, and T.

(*c*) Circles O and P, which both pass through the vertices of $\triangle ABC$, are equal.

Ex. 138. In $\odot$ O, there are three equal central angles, $\angle 1$, $\angle 2$, and $\angle 3$. The arc intercepted by $\angle 1 = 34°$. How large are the arcs intercepted by $\angle 2$ and $\angle 3$?

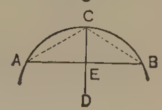
Ex. 139. On $\odot$ O, there are five equal arcs, each of which is intercepted by a central angle. One of these angles contains 31°. How large is each of the other central angles?

Ex. 140. In Ex. 139, the chords of the five arcs are drawn. One of these chords is 1 in. long. How long is each of the others?

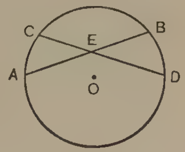


Ex. 141. In a $\odot$ O, radius ODC is $\perp$ chord AB at D. $AD = 2$ in., and $\overset{\frown}{AC} = 25°$.

(*a*) How long is AB? (*b*) How large is $\overset{\frown}{ACB}$? (*c*) How large is $\angle AOB$?



Ex. 142. If $AE = 2$ in., and $EB = 2$ in., and $CED \perp AB$, where does the center of the circle lie?

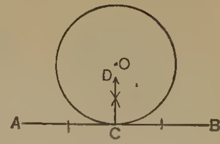


Ex. 143. In $\odot$ O, $AB = 5$ in., and is 2 in. from the center of the circle. How far is O from CD if $CD = 5$ in.?

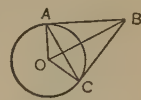
Ex. 144. In $\odot$ O, O is 3 in. from AB and also from CD. AB is 6 in. long. How long is CD?

Ex. 145. In O, $AB = 4$ in. and CD is 5 in. AB is 3 in. from O. What do you know about the distance of CD from O?

Ex. 146. If $\overset{\frown}{AC} = 30°$ and $\overset{\frown}{BD} = 40°$, how large is $\angle CEA$?



Ex. 147. The radius OC is 4 in.; $AB \perp OC$, at C. What do you know about the distance from O of every point on AB, except C?



Ex. 148. In the figure adjoining, if $\angle A$ and $\angle C$ are each right angles, and O is the center of the circle:

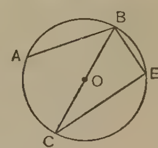
(*a*) How long is BC if $AB = 8$ in.?

(*b*) How large is $\angle OBC$ if $\angle ABO = 30°$?

Ex. 149. In the figure for Ex. 148, if $\angle AOC = 125°$, how large is $\overset{\frown}{AC}$?

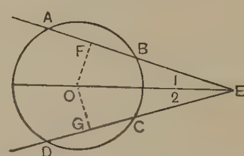
Ex. 150. In the figure for Ex. 148, if $\overset{\frown}{AC} = 140°$, how large is $\angle ABC$?

Ex. 151. In the figure for Ex. 148, if $\overset{\frown}{AC} = 140°$, how large is $\angle ACB$?



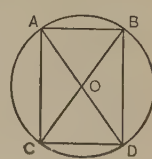
Ex. 152. In the adjoining figure, if $\overset{\frown}{AC} = 68°$, how large is $\angle ABC$?

Ex. 153. In the adjoining figure, if O is the center of the circle, how large is $\angle BEC$?



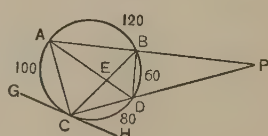
Ex. 154. If in $\odot O$, $\overset{\frown}{AD} = 100°$, and $\overset{\frown}{BC} = 50°$, how large is $\angle AED$?

Ex. 155. If, in the adjoining figure, $\angle 1 = \angle 2$; if $OF \perp AE$, and $OG \perp DE$; and if $AB = 5$ in., how long is CD?



Ex. 156. In the adjoining figure, if $AB \parallel CD$, and $\overset{\frown}{AC} = 88°$, how large is $\overset{\frown}{BD}$?

Ex. 157. If, also, $AC = 1$ in., how long is BD?



Ex. 158. In the adjoining figure:

(*a*) how large is $\angle CAB$?

(*b*) how large is $\angle AEC$?

(*c*) how large is $\angle ADC$?

(*d*) how large is $\angle APC$?

(*e*) how large is $\angle ACG$, if GH is a tangent?

Ex. 159. Compare chords AC and CD.

Ex. 160. What do you know about AD?

Ex. 161. Compare $\angle ACB$ and $\angle ADB$.

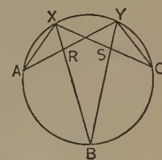
Miscellaneous Exercises

Ex. 162. AOB is a diameter of $\odot O$. C is any point of $\widehat{AB}$. D is the mid-point of $\widehat{BC}$ and E is the mid-point of $\widehat{AC}$. Prove $\angle DOE$ is a right angle. (*Suggestion.* Draw CO.)

Ex. 163. If point B bisects $\widehat{AC}$ of a circle, then $\angle A$ of $\triangle ABC$ equals $\angle C$.

Ex. 164. If the diagonals of an inscribed quadrilateral are equal, the quadrilateral is an isosceles trapezoid.

Ex. 165. Prove that two chords which are perpendicular to a third chord at its extremity are equal.



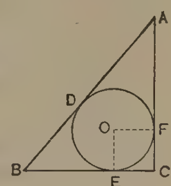
Ex. 166. If $\widehat{XA} = \widehat{YC}$ and $\widehat{BC} = \widehat{AB}$, prove $\triangle AXR \cong \triangle YCS$.

Ex. 167. In the figure for Ex. 166 draw AC cutting XB at M and YB at N. Prove $\triangle AXM \cong \triangle YCN$.

Ex. 168. Prove that the bisector of the angle between two tangents to a circle passes through the center of the circle.

Suggestions. Draw radii to the points of contact. Recall § 117, II.

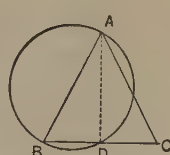
Ex. 169. Points A and B are on the diameter XY of circle O at equal distances from O. CA and DB are perpendicular to XY, meeting the semicircle at C and D respectively. Prove $ABDC$ is a rectangle.



Ex. 170. If a circle is inscribed in a right triangle, the sum of its diameter and the hypotenuse is equal to the sum of the legs of the triangle.

Ex. 171. Prove that the bisectors of the angles of a circumscribed quadrilateral pass through a common point.

Suggestion. Use Ex. 168.



Ex. 172. The circle drawn on one of the equal sides of an isosceles triangle as diameter bisects the base. (Figure at the right.)

Ex. 173. If AB and AC are the tangents from point A to the circle O, $\angle BAC = 2\ \angle OBC$.

Suggestions. 1. Draw OA. What relation does it bear to BC? 2. Compare $\angle BAO$ with $\angle OBC$.

OPTIONAL TOPICS

230. Optional topics occupy the pages 134 to 150 inclusive. The status and purpose of these topics has been described on page 86. It may be repeated here that *they are optional in the sense that they are not among the topics which are mentioned explicitly in the list of fundamental theorems or in the list of subsidiary theorems of either the National Committee or of the College Entrance Requirements Board.* This is sufficient official sanction for excluding them from a minimum course, but should not be interpreted as implying any disbelief in their educational value or mathematical interest and significance.

Topic A. **Theorems Relating to Two Circles.**

The definitions, theorems, and exercises included in this topic have appeared in most geometries. Most teachers would agree that it would be unfortunate to fail to bring at least the good and excellent pupils into contact with them. (Pages 135–138.)

Topic B. **Further Discussion of Loci.**

This is a more complete discussion of loci than is proposed explicitly for a minimum course, but embraces nothing more than has appeared in geometries for years, — except for improved presentation. The exercises and methods are highly instructive mathematically, and are both interesting and easy. They are provided as a challenge for at least the good and excellent pupils.

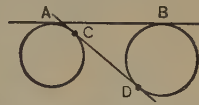
Topic C. **Further Discussion of Constructions.**

The exercises appearing in this group also have been found in geometries for years, — but unaccompanied often with the necessary instruction to enable pupils to acquire any skill in solving them. For this reason, they have been omitted by many classes, and are not mentioned explicitly in the reports of the committees. Of their educational value, as illustrations of the ideals and power of mathematics, there can be no question.

Optional Topic A

Theorems Relating to Two Circles

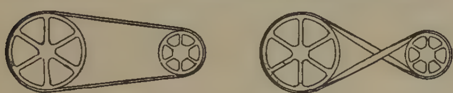
231. A straight line tangent to each of two circles is called a **common tangent of the circles**; as AB or CD.



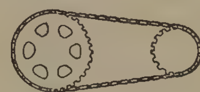
If the circles lie on opposite sides of CD, then CD is called a **common internal tangent** of the circles.

If the circles lie on the same side of AB, then AB is called a **common external tangent** of the circles.

The **length of the common tangent** of two circles is the length of the segment between the two points of contact.

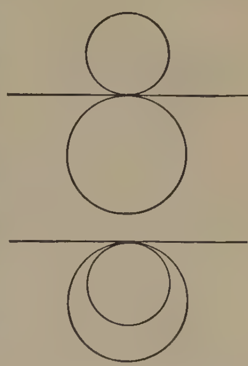


Belts around Pulleys



Chain around Wheels

232. Two circles are tangent to each other when they are tangent to the same straight line at the same point.



They are **tangent externally** if they lie on opposite sides of the common tangent.

They are **tangent internally** if they lie on the same side of the tangent.

Whenever tangent circles are involved in an exercise, draw their common tangent at their point of contact.

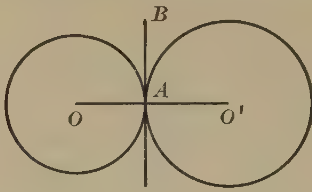
Ex. 174. Prove that the common internal tangents of two circles are equal. (There are two such tangents.)

Ex. 175. Prove that the common external tangents of two unequal circles are equal.

Plan. Extend the tangents until they meet.

PROPOSITION XXIII. THEOREM

233. *If two circles are tangent to each other, their line of centers passes through the point of contact.*



Hypothesis. ⓢ O and O' are both tangent to AB at A.
OO' is the line of centers.

Conclusion. Straight line OO' passes through A.

Plan. Prove OO' coincides with a straight line through O, A, and O'.

Proof: STATEMENTS	REASONS
1. Draw radii OA and $O'A$.	1. Why possible?
2. $\therefore OA \perp AB$; and $O'A \perp AB$.	2. § 198.
3. $\therefore \angle O'AB + \angle BAO = 1$ st. $\angle$.	3. Give the proof.
4. $\therefore O'AO$ is a st. line.	4. § 41.
5. $\therefore$ st. lines $O'AO$ and $O'O$ coincide.	5. Post. 1, § 50.
6. $\therefore O'O$ passes through A.	6. Since $O'AO$ does.

234. Cor. *If the distance between the centers of two circles equals the sum of their radii, the circles are tangent externally.*

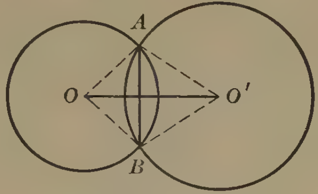
For then a point A can be taken on OO' so that OA = one radius and $O'A$ = the other radius. A perpendicular to OO' at A will then be tangent to each of the circles. Hence the circles are tangent. (§ 232.)

Ex. 176. If two circles are tangent internally, the line of centers passes through the point of contact.

Ex. 177. If a straight line be drawn through the point of contact of two circles which are tangent externally, terminating in their circumferences, the tangents at its extremities are parallel.

PROPOSITION XXIV. THEOREM

235. *If two circles intersect, the straight line joining their centers bisects their common chord at right angles.*

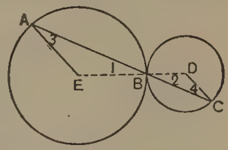


Hypothesis. Ⓢ O and O' intersect at A and B. AB is the common chord and OO' is the line of centers.

Conclusion. $OO' \perp AB$ and OO' bisects AB.

Suggestion. Draw OA, OB, $O'A$, and $O'B$. (Apply § 77.)

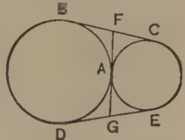
Ex. 178. If two circles O and O' intersect at points A and B, and if OO' intersects $\odot\, O$ at X and $\odot\, O'$ at Y, then X and Y are each equidistant from A and B.



Ex. 179. If a straight line be drawn through the point of contact of two circles which are tangent externally, terminating in the circles, the radii drawn to its extremities are parallel.

Note. The theorem is stated for two circles which are tangent externally. Investigate its truth for two circles which are tangent internally.

Ex. 180. If two circles are tangent to each other externally at point A, the tangents to them from any point in their common tangent which passes through A are equal.



Ex. 181. If two circles are tangent to each other externally at point A, the common tangent which passes through A bisects the other two common tangents.

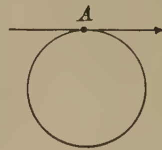
Ex. 182. Two circles are tangent externally at C. In one circle $\triangle ABC$ is inscribed, having one vertex at the point of contact of the circles. AC and BC are extended through C, meeting the other circle at D and E respectively. Prove $DE \parallel AB$.

Suggestion. Draw the common tangent through point C.

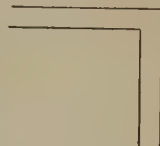
236. Theorems concerning tangents and tangent circles have unusually wide application in design.

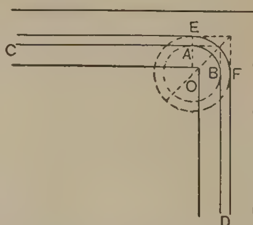
Direction is indicated by a straight line.

On a circle, the direction is constantly changing. It is convenient in both pure and applied mathematics to speak of *the direction of a curve at a point;* also it is agreed that this direction shall be the same as the direction of the tangent to the curve at the point. It is this fact which is used in a variety of ways.



If a road turns a corner as pictured, there is an abrupt change of direction. If a street car line runs along the road, such an abrupt change in direction in the tracks is impossible. For that reason, the arrangement of tracks indicated in the adjoining figure is employed. A car running from C towards D passes readily from CA to the arc AB, for on both the straight line and the arc the direction at A is the direction of line CA; similarly at B.





Ex. 183. Where must the center O be located in order that the circle will be tangent to both CA and BD?

Ex. 184. What kind of circles should circles AB and EF be?

Ex. 185. Determine how to construct the figures at the right. Construct such figures in circles with diameter 3 in.

Ex. 186. If AB is a common external tangent of two circles which touch each other externally at C, prove $\angle ACB$ is a right angle.

Ex. 187. Study the adjoining figure to determine how to construct it. Construct a figure like it, having the radius of the large circle 1 in. and that of the small circles $\frac{1}{2}$ in.

Suggestion. Draw the common tangent of the Ⓢ at C, meeting AB at D.

Note. Additional Exercises 60 to 64, page 287, can be done now.

Optional Topic B

Further Discussion of Loci

237. Method of attacking a locus problem. (See §§ 227, 228.)

1. Locate either freehand or by construction three or more points which satisfy the given condition. These points should suggest the *probable locus.*

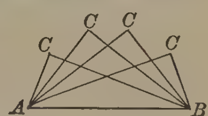
2. Draw the *probable locus* and try to prove that it is the *real locus.* To do this, try to prove either (*a*) and (*b*) below, or else (*a*) and (*c*):

(*a*) Every point on the locus satisfies the given condition.

(*b*) Every point which satisfies the given condition lies on the locus. (This is the *converse* of (*a*).)

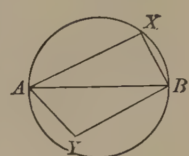
(*c*) Every point not on the locus does not satisfy the given condition. (This is the *opposite* of (*b*).)

238. Illustrative Problem. Determine the locus of the vertex of the right angle of a right triangle having a given segment as hypotenuse.



Solution. 1. Let $\triangle ABC$ be right triangles having the hypotenuse AB and rt. $\angle$ at C.

2. This figure suggests that the points C lie on a circle having AB as diameter.



3. Assume that the locus is the circle having AB as diameter.

(*a*) Is every $\triangle AXB$, *where X is any point on the circle,* a rt. $\triangle$?

Answer. Yes, since $\angle AXB$ is a rt. $\angle$, by § 213.

(*b*) Is every $\triangle AYB$, *where Y is any point not on the circle,* an oblique $\triangle$?

Answer. Yes, since every $\angle AYB$ is either acute or obtuse, according as Y lies outside or inside of the circle.

4. *Hence the locus of the vertex of the right angle of a right triangle having a given segment as hypotenuse is a circle drawn on the hypotenuse as diameter.*

239. In § 237, the *opposite* of statement (*b*) is:

(*d*) every point which does not satisfy the given condition does not lie on the locus.

If statements (*a*) *and* (*b*) *are known, then* (*c*) *and* (*d*) *can be proved easily by the indirect method of proof.*

If statements (*a*) *and* (*c*) *are known, then* (*b*) *and* (*d*) *can be proved easily by the indirect method of proof.*

Every locus theorem, therefore, is a short way of expressing all four of these statements for a given condition.

240. Summary of fundamental loci.

1. The locus of points at a given distance d from a given point O is the circle drawn with O as center and d as radius. (§ 225.)

2. The locus of points at a given distance d from a fixed line l (of indefinite length) is the pair of parallels to l at the distance d from it. (Ex. 122.)

3. The locus of points equidistant from two parallel lines is the line parallel to them and midway between them. (Ex. 124.)

4. The locus of points equidistant from two given points is the perpendicular-bisector of the segment joining the points.

5. The locus of points equidistant from the sides of an angle and within the angle is the bisector of the angle.

Cor. The locus of points equidistant from two intersecting straight lines is the set of bisectors of their included angles. These bisectors form two straight lines.

6. The locus of the vertex of the right angle of a right triangle which has a given hypotenuse is a circle drawn upon the hypotenuse as diameter. (§ 238.)

Ex. 188. Assuming theorems (*a*) and (*b*) of § 237 for locus 1, prove the theorems (*c*) and (*d*), using the indirect method of proof.

Ex. 189. (*a*) Drawing a line AB, and a segment s, 1 inch long, construct the locus of points 1 inch from AB.

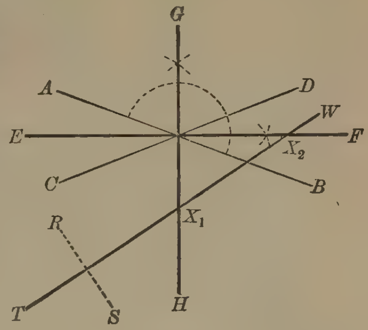
(*b*) Draw two parallels. Construct the locus of points equidistant from these parallels.

241. Intersection of loci. Sometimes it is specified that a point shall satisfy each of two given conditions. Each condition determines a locus for the point. The required point then must lie at the intersection of the two loci.

Illustrative problem. Find all points which are (*a*) equidistant from two intersecting lines and (*b*) also equidistant from two fixed points.

Given intersecting lines AB and CD and points R and S.

Required to find all points which are equidistant from AB and CD and also equidistant from R and S.



Solution. 1. The locus of points equidistant from AB and CD is the set of bisectors of the angles included by them. (Lines EF and GH.)

2. The locus of points equidistant from R and S is the perpendicular bisector of RS. (Line TW.)

3. The required points will be at the intersection of TW with EF and GH.

Discussion. 1. There may be two points; as X_1 and X_2.

2. There may be only one point, however, for TW may be parallel to EF or GH.

3. There must always be at least one point, for TW cannot be parallel to both EF and GH.

4. There may be a whole line full of points, for TW may coincide with EF or GH.

NOTE 1. Draw, free hand, figures to represent the possibilities 2 and 4.

NOTE 2. (*a*) Follow closely the form of solution used above. (*b*) Construct the figures described in the solution.

Ex. 190. In a given line, find all points which are equidistant from two given points.

Ex. 191. In a given line, find all points which are equidistant from two given intersecting lines.

Ex. 192. In a given circle, find all points which are equidistant from two given parallel lines.

Ex. 193. Find all points which are equidistant from two given points and also at a given distance from a given point.

Ex. 194. Find all points which are equidistant from two given points and also at a given distance from a given line.

Ex. 195. Find all points which are equidistant from two given points and also equidistant from two given parallels.

Ex. 196. Find all points which are equidistant from two given parallels and also at a given distance from a given point.

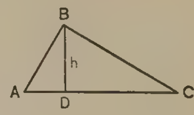
Ex. 197. Find all points which are at a given distance from a given line and also at another given distance from a given point.

Ex. 198. Find all points which are equidistant from two parallels and also equidistant from two intersecting lines.

Ex. 199. Find all points which are equidistant from two intersecting lines and at a given distance from a given point.

Ex. 200. What is the locus of the vertex of a triangle whose base lies in a given straight line if the altitude to the base is a given segment?

Suggestion. If AC is the base and $BD = h$, the given altitude, then B must be at distance h from AC. What then is the locus of B?



Ex. 201. What is the locus of the center of a circle which shall be tangent to a given line and have a given radius?

Ex. 202. What is the locus of points at a given distance from a given circle?

Suggestion. The distance is measured along a line between the point and the center of the circle.

Ex. 203. What is the locus of the center of a circle which has a given radius and passes through a given fixed point?

Ex. 204. What is the locus of the center of a circle which shall be tangent to each of two parallel lines?

Ex. 205. What is the locus of the mid-points of all chords of a circle that have a given length?

Ex. 206. What is the locus of the points such that the tangents from the points to a given circle shall have a given length?

Ex. 207. What is the locus of the mid-points of all parallel chords of a circle?

Ex. 208. What is the locus of the mid-points of all segments drawn from one vertex of a triangle and terminated by the opposite side?

Optional Topic C

Further Discussion of Construction Problems

242. Analysis of construction problems.

1. Draw a figure which represents the desired figure.

(*a*) Make this figure general. For example, if a triangle is to be drawn, do not draw a right or an isosceles triangle unless such a triangle is specified.

(*b*) Remember that this is not the final figure and that the parts in it are not necessarily the given parts.

2. Mark with heavy lines or with colored lines the parts which are known and also those which may be readily determined from the known parts by fundamental constructions.

For example, if a known line is bisected, then each of the halves is known.

3. Try to determine some part of the figure which can be constructed by known methods. Usually this is a triangle. This part can usually be made the basis for the rest of the construction.

In this connection, remember the first five fundamental triangle constructions given in § 224. (Prop. IV, Book I, and Exercises 109, 111, 113, 115, of Book II.)

4. Try to determine how the remaining parts can be obtained from the figure constructed in Step 3.

5. Make the construction, following the plan decided upon in Steps 3 and 4.

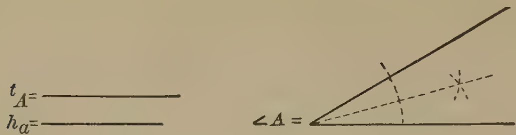
6. Prove that the resulting figure satisfies the conditions of the problem.

7. Discuss the construction, determining when the construction is possible, and when it is not possible.

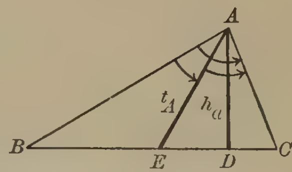
This plan will be illustrated in the solution of the following problem. Before reading it, review the notations described in § 223, page 126. Then read the solution, and, afterwards, solve the exercises which follow.

Illustrative Problem. Construct a triangle, having given an angle, its bisector, and the altitude to the opposite side.

Given



Required to construct $\triangle ABC$.



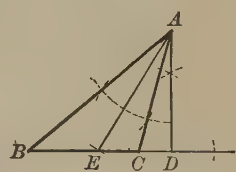
Analysis. 1. Let $\triangle ABC$, with $AD \perp BC$ and AE bisecting $\angle A$, represent the required figure.

2. The known parts are marked with heavy lines, including $\angle BAE = \angle EAC = \frac{1}{2}\angle A$.

3. $\triangle ADE$ is a rt. $\triangle$ with a known leg $(= h_a)$ and known hypotenuse $(= t_A)$. Hence $\triangle ADE$ can be constructed. (Ex. 115.)

4. B is on DE extended and $\angle BAE = \frac{1}{2}\angle A$.

5. C is on ED extended and $\angle EAC = \frac{1}{2}\angle A$.



Construction. 1. Construct rt. $\triangle ADE$ with leg $= h_a$ and hypotenuse $= t_A$.

2. Extend DE in both directions.

3. Bisect $\angle A$, and construct AB, making $\angle EAB = \frac{1}{2}\angle A$; let AB meet DE extended at B.

4. Construct AC, making $\angle EAC = \frac{1}{2}\angle A$, and meeting ED at C.

5. Then $\triangle ABC$ is the required triangle.

Proof. 1. $\angle BAC$ = given $\angle A$, since it equals $2(\frac{1}{2}\angle A)$. Const.

2. AD = given h_a and is an altitude, since $\angle D$ = rt. $\angle$. Const.

3. AE = given t_A and is the bisector of $\angle A$. Const.

Discussion. The construction is impossible if $t_A < h_a$.

Note. Observe that the final triangle may appear quite different from the triangle drawn for the first step in the analysis.

Ex. 209. (*a*) Make the construction for the preceding problem when $\angle A = 60°$, $t_A = 2$ in., and $h_a = 1$ in. Measure the side a.

(*b*) Repeat the construction, changing $\angle A$ to 90°. Measure a.

(*c*) What happens to side a when $\angle A$ increases, if t_A and h_a remain unchanged?

Ex. 210. Construct a triangle having given two sides and the altitude to one of them.

Given b, c, and h_c.

Required to construct $\triangle ABC$.

Make the analysis, construction, proof, and discussion, following *closely* the model solution on page 144.

Note. After completing the general exercise, or even as a substitute for it or preparation for it, have the construction made for selected values of b, c, and h_c (as in Ex. 209) and ask for the length of a. For example, let $b = 2$ in., $c = 2.5$ in., and $h_c = 1$ in. Then keeping b and c fixed, but increasing h_c, have the construction repeated, and again measure side a. Finally ask, what happens to a when h_c increases, and when b and c remain unchanged. This is a means of introducing the notion of functional relationship into geometry.

Ex. 211. Construct $\triangle ABC$, given a, h_a, and m_a.

Ex. 212. Construct $\triangle ABC$, having given h_b, m_b, and a.

Ex. 213. Construct $\triangle ABC$, having given h_c, m_c, and $\angle B$.

Ex. 214. Construct $\triangle ABC$, having given c, h_a, and b.

Ex. 215. Construct $\triangle ABC$, having given h_b, c, and $\angle C$.

Ex. 216. Construct $\triangle ABC$, having given b, h_b, and $\angle A$.

Ex. 217. Construct $\triangle ABC$, having given h_b, t_B, and c.

Ex. 218. Construct an isosceles triangle having given the base angle and the altitude to the base.

Ex. 219. Construct an isosceles triangle having given one side and the altitude to one of the sides.

Ex. 220. Construct an isosceles triangle having given the base and the altitude to the one of the equal sides.

Ex. 221. Construct a right triangle having given the altitude upon the hypotenuse and one of the legs.

Ex. 222. Construct a right triangle having given the altitude upon the hypotenuse and one of the acute angles.

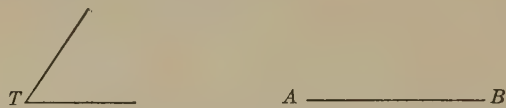
Note. Additional Exercises 65 to 83, p. 287 can be done now.

243. Systematic use of analysis is attributed to Plato. The method of analysis has been described as one of the four great forward steps in mathematics. Plato also introduced the restriction that constructions in elementary geometry shall be made by ruler and compass alone, and it is for this reason that many constructions are impossible. *Among them is that of trisecting an angle.*

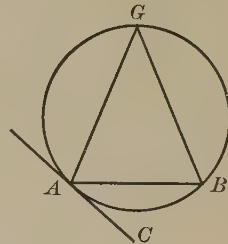
PROPOSITION XXV. PROBLEM

244. *Upon a given segment as chord, construct an arc of a circle such that every angle inscribed in the arc shall equal a given angle.*

Given angle T and segment AB.



Required to construct an arc upon AB as chord such that every angle inscribed in the arc shall equal $\angle T$.



Analysis. 1. Assume that $\overset{\frown}{AGB}$ is the required arc, so that $\angle AGB = \angle T$; also assume AC is tangent to $\odot$ AGB, so that $\angle CAB$ also $= \angle T$.

2. Since the center is equidistant from A and B, one locus for it is the $\perp$-bis. of AB (§ 227).

3. Since AC is tangent at A, another locus of the center is the line perpendicular to AC at A.

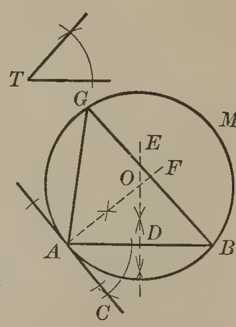
4. This suggests the following construction.

Construction. 1. Construct $\angle BAC = \angle T$.

2. Construct $DE \perp AB$ at its midpoint.

3. Construct $AF \perp AC$ at A, intersecting DE at O.

4. Construct a circle with center O and OA as radius.



Statement. $\overset{\frown}{AMB}$ is the required arc.

Proof. (The pupil should now prove that any $\angle AGB = \angle T$.)

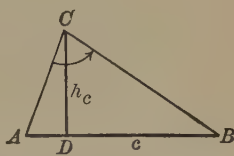
Discussion. The construction is possible if $\angle T < 180°$.

245. Cor. (Locus 7.) If A and B are two fixed points and X is a point such that $\angle AXB$ is equal to a given angle, the locus of X is the arc of a circle constructed on AB as chord such that every angle inscribed in it equals the given angle.

Illustrative problem. Construct $\triangle ABC$ having given c, h_c, and $\angle C$.

Given

Required to construct $\triangle ABC$.

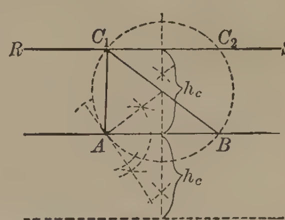


Analysis. 1. Let $\triangle ABC$ represent the desired triangle, the known parts being marked by heavy lines.

2. Line c can be drawn, thus locating definitely points A and B.

3. Point C is at the distance h_c from c. It therefore lies on one of two parallels to c at the distance h_c from c. (Locus 2, § 240.)

4. Point C is such that $\angle ACB$ must equal $\angle C$. It therefore lies on the arc constructed on AB as chord, the inscribed angles of which equal $\angle C$. (Locus 7, § 245.)



Construction is made so as to obtain the loci mentioned in Steps 3 and 4. The circle cuts the line RS at two points C_1 and C_2. $\triangle AC_1B$ and $\triangle AC_2B$ each satisfy the given conditions.

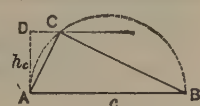
Proof and Discussion left to the pupil.

Ex. 223. Given the base angle and the altitude to the base of an isosceles triangle, construct the triangle.

Ex. 224. Construct an isosceles triangle having given the base and the radius of the circumscribed circle.

Ex. 225. Construct a rhombus having given its base and altitude.

Ex. 226. Construct a right triangle having given the hypotenuse and the length of the altitude upon it.

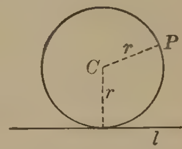


Ex. 227. Construct an isosceles triangle having given the base and the angle opposite the base.

Ex. 228. Construct a triangle having given the base, the altitude, and the radius of the circumscribed circle.

Ex. 229. Construct a triangle having given a side, an adjacent angle, and the radius of the circumscribed circle.

Ex. 230. Through a given point construct a circle with a given radius which shall be tangent to a given line.



Analysis. 1. Let the circle with the center C pass through P and be tangent to line l.

2. C is r distant from P. Hence it must lie on a circle having P as center and r as radius.

3. C is r distant from l. Hence C must lie on one of two parallels to l at the distance r from l.

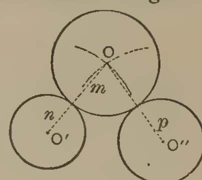
4. C must be at the intersection of these two loci.

Construction, Proof, and Discussion left to the pupil.

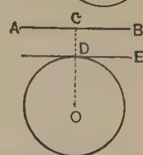
Ex. 231. Construct a circle with a given radius which shall be tangent to each of two intersecting lines.

Ex. 232. Construct a circle which shall be tangent to each of two intersecting lines, tangent to one of them at a given point.

Ex. 233. Construct a circle through a given point not in a given line which shall be tangent to the given line at a given point in the line.



Ex. 234. Construct a circle having a given radius which shall be tangent to each of two given circles.

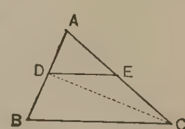


Ex. 235. Construct a tangent to a circle which will be parallel to a given line.

Suggestion. Make an analysis based on the adjoining figure.

Ex. 236. Through a given point within a circle, construct a chord equal to a given chord.

Is there any restriction on the location of the point?



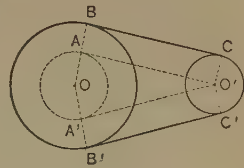
Ex. 237. Construct a parallel to the side BC of $\triangle ABC$ meeting AB and AC at D and E respectively, so that DE will equal EC.

Note. Additional Exercises 84 to 92, page 289, can be done now.

Practical Applications

Ex. 238. Construct the common external tangents of two circles.

Analysis. 1. Let the circles be assumed unequal. Let the adjoining figure represent the desired figure.



2. Evidently $ABCO'$ is a ▭.
3. $\therefore AB = CO'$.
4. $\therefore AO = OB - CO'$.
5. Also, AO' and $A'O'$ are tangents to circle AA'.

Construction. 1. Construct a circle with radius equal to the difference between the radii of the two circles, and concentric with the larger circle.

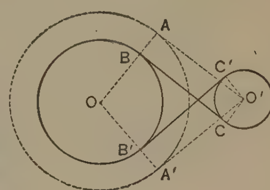
2. Draw tangents to this circle from the center of the smaller circle, meeting the constructed circle at points A and A'.

3. Draw OA and OA' meeting the large circle at B and B'.

Complete the construction.

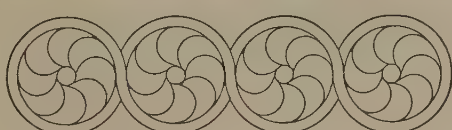
Give the Proof and the Discussion.

Ex. 239. Construct the common internal tangents of two unequal circles.

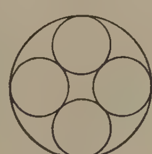


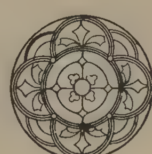
Ex. 240. A circular cylinder head 12 in. in diameter is to have holes bored in it for 12 bolts, equally spaced around the edge, with their centers $1\frac{1}{2}$ in. from the edge. Make a scale drawing of the cylinder head ($\frac{1}{4}'' = 1''$) and mark the centers for the 12 bolt holes.

Ex 241. Determine how to construct the unit which is repeated in the design below. Construct it in a circle of 3-in. radius.

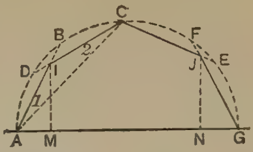


Ex. 242. Determine how to construct the left-hand figure. Notice that it is the basis for the artistic design at the right. Construct the left-hand figure in a circle of 4-in. diameter.





Ex. 243. The adjoining figure indicates a form of mansard roof. The chords, AB, CD, CE, and FG are all equal.

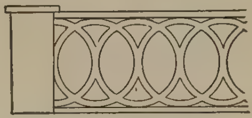


Construct such a roof outline for a building in which AG is 30 ft. and the distance IM is 10 ft. (Let 1 in. = 5 ft.)

Is $AI = IC = CJ = JG$?

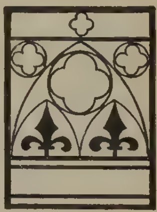
Is the line $IJ \parallel AG$?

Ex. 244. Construct between two parallel lines a set of circular rings like those in the design below.

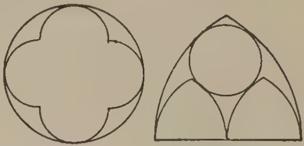


DESIGN FOR ORNAMENTAL STONEWORK ON A BRIDGE

Ex. 245. The adjoining design is a panel for ornamental ironwork on a bridge.

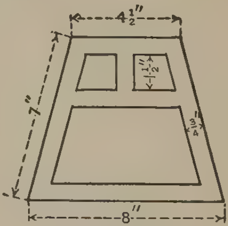


Determine how to construct the fundamental units of the design, units (a) and (b.)

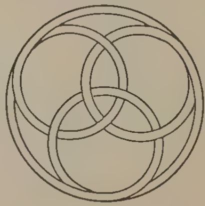


UNIT (a) UNIT (b)

Ex. 246. Construct full size the pattern for the faces of a mission lamp as shown in the adjoining figure, using the dimensions indicated.



Ex. 247. Construct a figure like the one adjoining, using for the smallest equal circles from which it is constructed the radius 1 in. Notice that the circles are tangent circles.



BOOK III

PROPORTION — SIMILAR POLYGONS

246. The **ratio** of one number to another is the quotient of the first divided by the second. It is a means of comparing the numbers.

Thus, the ratio of 3 to 4 is $\frac{3}{4}$; of a to b is $\frac{a}{b}$.

The first number is made the numerator and is called the **antecedent**; the second is made the denominator and is called the **consequent**. Together, the numbers are called the **terms of the ratio**.

Since a ratio is a fraction, it is added, subtracted, divided, etc. as any fraction.

To aid the pupils who may have forgotten how to handle fractions, a brief review of fractions and elementary algebra is given on pages 311 to 314. A few days spent in review will be time well spent.

247. Concrete quantities (like angles, segments, coal, etc.) are compared only when they are **of the same kind**.

Thus, an angle is not compared with a segment.

The ratio of two concrete quantities of the same kind is the ratio of their measure in terms of a common unit. (§ 204, *a*.)

Thus, the ratio of 350 lb. of coal and 1 ton is $\frac{350}{2000}$ or $\frac{7}{40}$.

Ex. 1. Express the following ratios in their simplest form.

(*a*) 3 to 9. (*c*) $5x$ to $2x$. (*e*) $\frac{5}{8}$ to $\frac{3}{16}$. (*g*) 25 to 375.
(*b*) 12 to 2. (*d*) $6a^2$ to $15a^3$. (*f*) $\frac{2}{15}$ to $\frac{1}{3}$. (*h*) $a^2 - b^2$ to $a + b$.

Ex. 2. What is the ratio of: (*a*) a right angle to a straight angle? (*b*) a right angle to the perigon? (*c*) one angle of an equilateral triangle to the sum of all the angles of the triangle? (*d*) one side of a square to the perimeter of the square?

Ex. 3. The ratio of the height of a tree to the length of its shadow on the ground is 17 : 20. Find the height of the tree if the length of the shadow is 110 feet.

248. A **proportion** is a statement that two ratios are equal; as, $\frac{a}{b} = \frac{c}{d}$, or $a : b = c : d$.

This proportion is read "a is to b as c is to d."

Thus, 1, 3, 5, and 15 form a proportion since $\frac{1}{3} = \frac{5}{15}$.

This means that 1 bears to 3 the same relation that 5 bears to 15.

The first and fourth terms of a proportion are called the **extremes**, and the second and third terms, the **means**.

a and d above are the extremes and b and c are the means; a and c are the antecedents, and b and d are the consequents.

Ex. 4. Select four numbers which form a proportion like the arithmetical illustration in § 248.

Ex. 5. (*a*) Is $\frac{3}{4} = \frac{5}{6}$? (*b*) Is $\frac{2}{4} = \frac{6}{8}$? (*c*) Is $\frac{3}{9} = \frac{4}{16}$?

Ex. 6. Find the value of the literal number in each of the following proportions. (If necessary, review pages 311 to 314.)

(*a*) $\frac{x}{4} = \frac{5}{8}$. (*c*) $\frac{6}{16} = \frac{c}{8}$. (*e*) $\frac{2 + x}{2} = \frac{5}{3}$.

(*b*) $\frac{10}{y} = \frac{2}{3}$. (*d*) $\frac{1}{8} = \frac{3}{z}$. (*f*) $\frac{3 + t}{4 - t} = \frac{5}{2}$.

Ex. 7. Find the value of x in each of the following proportions.

(*a*) $\frac{a}{b} = \frac{c}{x}$. (*b*) $\frac{a}{3b} = \frac{x}{2c}$. (*c*) $\frac{r^2}{s} = \frac{rx}{t}$. (*d*) $\frac{mn}{p} = \frac{cn}{x}$.

(*e*) $\frac{\overline{AB}}{\overline{CD}} = \frac{\overline{EF}}{x}$. (*f*) $\frac{\overline{AB}}{\overline{AC}} = \frac{\overline{AD}}{x}$. (*g*) $\frac{\overline{AB}}{x} = \frac{x}{\overline{CD}}$.

249. The **fourth proportional** to three numbers a, b, and c is the number x in the proportion $a : b = c : x$.

Thus, the fourth proportional to 2, 3, and 4 is the number x in $\frac{2}{3} = \frac{4}{x}$.

$$\therefore 2x = 12, \text{ or } x = 6.$$

Note. The numbers *must* be placed in the proportion *in the order in which they are given* as in the illustrative example. Observe the results in Ex. 8, *a*, *b*, and *c*.

Ex. 8. Find the fourth proportional to:

(*a*) 2, 3, and 4.
(*b*) 3, 2, and 4.
(*c*) 4, 3, and 2.
(*d*) 25, 15, and 10.
(*e*) $3a$, $2b$, and c.
(*f*) r, rs, and s.

250. *In a proportion, the product of the extremes is equal to the product of the means.*

If $\frac{a}{b} = \frac{c}{d}$, then $\cancel{b}d \cdot \frac{a}{\cancel{b}} = b\cancel{d} \cdot \frac{c}{\cancel{d}}$, or $ad = bc$. (Ax. 5, § 49.)

Example. Since $\frac{2}{3} = \frac{6}{9}$; then 2×9 should $= 3 \times 6$. Does it?

251. A mean proportional between two numbers a and b is the number x in the proportion $a : x = x : b$.

Example. A mean proportional between 2 and 3 is x in:

$$\frac{2}{x} = \frac{x}{3}.$$

$$\therefore x^2 = 6, \text{ or, } x = \sqrt{6}.$$

$$\therefore x = \pm 2.4490.$$

See page 312.

There are then *two mean proportionals between two given numbers. The positive one is implied when "the" mean proportional is specified.*

The mean proportional between two numbers is the square root of their product.

For the mean proportional between a and b is x in the proportion

$$\frac{a}{x} = \frac{x}{b}. \quad \therefore x^2 = ab, \text{ or } x = \sqrt{ab}.$$

Ex. 9. Find the mean proportional between:

(*a*) 75 and 12. (*c*) 2 and 5. (*e*) $6a$ and $30a$.
(*b*) $2a$ and $8a$. (*d*) 4 and 12. (*f*) $5m$ and $20m$.

252. *If the product of two numbers is equal to the product of two other numbers, the first pair may be made the means of a proportion of which the other pair are the extremes.*

If $mn = xy$, then $\frac{mn}{xn} = \frac{xy}{xn}$, or $\frac{m}{x} = \frac{y}{n}$. (Ax. 6, § 49.)

Example. Since $3 \times 8 = 6 \times 4$, $\frac{3}{6}$ should equal $\frac{4}{8}$. Does it?

Also, $\frac{6}{8}$ should equal $\frac{3}{4}$. Does it?

Also, $\frac{6}{3} = \frac{8}{4}$. Does it?

253. *If the antecedents of a proportion are equal, then also the consequents are equal; if the consequents are equal, then also the antecedents are equal.*

If $a:x = a:y$, then $ay = ax$ (§ 250), and $\therefore\ y = x$. (Ax. 6, § 49.)

254. *If three terms of one proportion are equal respectively to the three corresponding terms of another proportion, the remaining terms also are equal.*

If $\frac{a}{x} = \frac{c}{d}$ and $\frac{a}{y} = \frac{c}{d}$, then $\frac{a}{x} = \frac{a}{y}$. $\therefore\ x = y$. (§ 253.)

255. *In any proportion, the terms are in proportion by* **alternation:** *that is, the first is to the third as the second is to the fourth.*

If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$. (§ 250.) $\therefore\ \frac{a}{c} = \frac{b}{d}$. (§ 252.)

256. *In any proportion, the terms are in proportion by* **inversion:** *that is, the second is to the first as the fourth is to the third.*

If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$. (§ 250.) $\therefore\ \frac{b}{a} = \frac{d}{c}$. (§ 252.)

257. *In any proportion, the terms are in proportion by* **addition:** *that is, the first plus the second is to the second as the third plus the fourth is to the fourth.*

If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} + 1 = \frac{c}{d} + 1$, or $\frac{a+b}{b} = \frac{c+d}{d}$. (Ax. 3, § 49.)

258. *In any proportion, the terms are in proportion by* **subtraction:** *that is, the first minus the second is to the second as the third minus the fourth is to the fourth.*

If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} - 1 = \frac{c}{d} - 1$, or $\frac{a-b}{b} = \frac{c-d}{d}$. (Ax. 4, § 49.)

259. *In any proportion, the terms are in proportion by* **addition** *and* **subtraction:** *that is, the first plus the second is to the first minus the second as the third plus the fourth is to the third minus the fourth.*

If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$. (Divide the result of § 257 by that of § 258.)

Ex. 10. Since $4 \times 5 = 2 \times 10$, write four proportions which involve 4, 5, 2, and 10. (Use § 252.)

Ex. 11. If $mn = xy$, write four proportions which involve m, n, x, and y.

Ex. 12. Write each of the following proportions by alternation:

(*a*) $\frac{2}{3} = \frac{10}{15}$; (*b*) $\frac{3}{5} = \frac{n}{r}$; (*c*) $\frac{r}{s} = \frac{x}{y}$; (*d*) $\frac{AB}{CD} = \frac{MN}{XY}$.

Ex. 13. Write the four proportions of Ex. 12 by inversion.

Ex. 14. Write the proportion $\frac{a}{b} = \frac{c}{d}$ by inversion, and then write the result by alternation.

Ex. 15. Write the four proportions of Ex. 12 by addition.

Ex. 16. Write the proportion $\frac{a}{b} = \frac{c}{d}$:

(*a*) By addition, and the resulting proportion by inversion.
(*b*) By inversion, and the resulting proportion by subtraction.
(*c*) By subtraction, and the resulting proportion by alternation.
(*d*) By alternation, and the resulting proportion by addition.

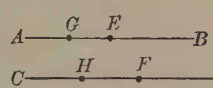
260. When dealing with ratios and proportions of *geometrical magnitudes*, we replace the magnitudes themselves by their measures in terms of common units of measure. Hence, the theorems given in § 250 to § 259, all apply to the proportions encountered in geometry.

Thus, if AB, CD, EF, and GH are four segments such that

$$\frac{AB}{CD} = \frac{EF}{GH}, \text{ then } AB \times GH = CD \times EF.$$

This means that the product of the numerical measures of AB and GH equals the product of the numerical measures of EF and CD.

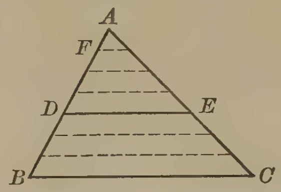
261. If $AE = EB$ and $CF = FD$, $\frac{AE}{EB} = \frac{CF}{FD}$ since each ratio equals 1. If G bisects AE and H bisects CF, then $\frac{AG}{GB} = \frac{1}{3}$ and also $\frac{CH}{HD} = \frac{1}{3}$; hence $\frac{AG}{GB} = \frac{CH}{HD}$. G and H are said to divide AB and CD proportionally.



Two line-segments are **divided proportionally** when the segments of one have the same ratio as the corresponding segments of the other.

PROPOSITION I. THEOREM

262. *A parallel to one side of a triangle, intersecting the other two sides, divides the other two sides proportionally.*



Hypothesis. In $\triangle ABC$, $DE \parallel BC$, meeting AB at D and AC at E.

Conclusion. $\dfrac{AD}{DB} = \dfrac{AE}{EC}$.

CASE I. When AD and DB are commensurable.

Proof: STATEMENTS	REASONS
1. Let $AD = 4\ AF$; and $DB = 3\ AF$.	1. They are commensurable.
2. $\therefore \dfrac{AD}{DB} = \dfrac{4}{3}$.	2. § 247.
3. $\parallel$s to BC from the division pts. on AB divide AE into 4 and EC into 3 equal parts.	3. § 141, p. 73.
4. $\therefore \dfrac{AE}{EC} = \dfrac{4}{3}$.	4. § 247.
5. $\therefore \dfrac{AD}{DB} = \dfrac{AE}{EC}$.	5. Ax. 1, § 49.

CASE II. When AD and DB are incommensurable.

The theorem is true for the incommensurable case also. The proof is given in § 399.

Ex. 17. (*a*) If, in the figure of § 262, AD is $\frac{1}{3}$ of BD, what is the ratio of AE to EC?

(*b*) If $AD = 5$ times DB, compare AE and EC.

(*c*) If $AD = DB$, compare AE and EC.

263. Cor. 1. Since $AD : DB = AE : EC$, then

$$\frac{AD + DB}{DB} = \frac{AE + EC}{EC}, \text{ or } \frac{AB}{DB} = \frac{AC}{EC}. \quad (\S\ 257)$$

That is, *one side is to its lower segment as the other side is to its lower segment.*

264. Cor. 2. Since $\frac{AD}{DB} = \frac{AE}{EC}$, then $\frac{DB}{AD} = \frac{EC}{AE}$. Why?

$$\therefore \frac{DB + AD}{AD} = \frac{EC + AE}{AE}, \text{ or } \frac{AB}{AD} = \frac{AC}{AE}. \quad \text{Why?}$$

That is, *one side is to its upper segment as the other side is to its upper segment.*

265. Other proportions can be derived from the proportions obtained in §§ 262–264 by making allowable changes in them.

From § 262, $AD : AE = DB : EC$. Why?
From § 263, $BD : AB = EC : AC$. Why?
From § 264, $AD : AB = AE : AC$. Why?

266. For convenience, reference may be made to any of the proportions developed in §§ 262–265 by quoting the authority:

A parallel to one side of a triangle divides the other two sides proportionally.

Ex. 18. Express each of the proportions of § 265 as was done (in italics) in § 263 and § 264.

Ex. 19. (*a*) If $AD = 8$ in., $DB = 5$ in., and $EC = 6$ in., find AE.
(*b*) If $DB = 7$ in., $AE = 10$ in.; and $EC = 9$ in., find AD.
(*c*) If $AD = 8$ in., $AE = 10$ in.; and $EC = 9$ in., find DB.

Ex. 20. (*a*) If $AB = 12$ in., $AC = 15$ in., and $AE = 6$ in., find AD.
(*b*) If $AB = 10$ in., $AD = 4$ in., and $AE = 3$ in., find AC.
(*c*) If $AB = m$ in., $AD = r$ in., and $AC = s$ in., find AE.

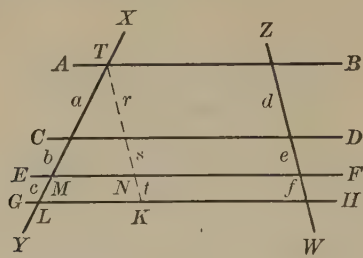
Ex. 21. Make the following construction, to find one third of a given segment AC.

1. Draw a line AX, making an acute $\angle XAC$.
2. On AX, take a short segment AD, and also $DB = 2\,AD$.
3. Draw BC, and through D, construct $DE \parallel BC$, meeting AC at E.
4. Prove $AE = \frac{1}{3} AC$.

Note. Additional Exercises 1 to 10, page 289, can be studied now.

PROPOSITION II. THEOREM

267. *Parallel lines intercept proportional segments on all transversals.*



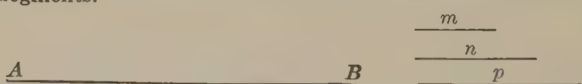
Hypothesis. Parallels AB, CD, EF, and GH intercept segments, a, b, and c, on XY and d, e, and f on ZW, respectively.

Conclusion. $$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}.$$

Suggestions. 1. Draw $TK \parallel ZW$. Let the $\parallel$s intercept the segments r, s, and t on TK.

2. Compare r, s, and t, with d, e, and f, respectively.
3. In $\triangle TKL$, compare $a : r$ with $TL : TK$.
4. In $\triangle TKL$, compare $c : t$ with $TL : TK$.
5. In $\triangle TMN$, compare $a : r$ with $b : s$.
6. Then compare $a : r$, $c : t$, and $b : s$.
7. Complete the proof, using the facts obtained in Step 2.

Ex. 22. Divide a segment into parts proportional to any number of given segments.



Given segment AB, and segments m, n, and p.

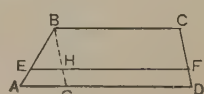
Required to divide AB into segments x, y, and z, so that

$$\frac{x}{m} = \frac{y}{n} = \frac{z}{p}.$$

Suggestion. Base the construction on Proposition II.

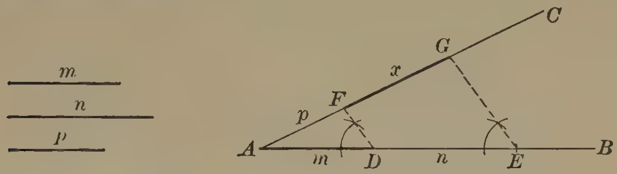
Ex. 23. A line drawn parallel to the bases of a trapezoid, and intersecting the non-parallel sides, divides the non-parallel sides proportionally.

Prove $BE : EA = CF : FD$.



PROPOSITION III. PROBLEM

268. *Construct the fourth proportional to three given segments.*



Given segments m, n, and p.

Required to construct the fourth proportional to m, n, and p.

Analysis. 1. Let x represent the fourth proportional.

Then $\qquad m : n = p : x.$

2. This suggests the following construction.

Construction. 1. On side AB of a convenient angle, $\angle BAC$, take $AD = m$, and $DE = n$; on AC, take $AF = p$.

2. Draw DF and construct $EG \parallel DF$, meeting AC at G.

Statement. FG is the fourth proportional to m, n, and p.

[Proof to be given by the pupil.]

Ex. 24. Construct the fourth proportional to segments which are 2 in., 1 in., and 3 in. in length. Measure the resulting segment.

Ex. 25. In § 268, if m and p remain constant (unchanged) and n is made longer, what happens to x?

Ex. 26. Construct x so that $m : n = n : x$, where m and n are the segments given above, or any given segments.

Ex. 27. Construct x so that $x = \dfrac{mn}{p}$.

Suggestion. If $x = \dfrac{mn}{p}$, then $xp = mn$. $\quad \therefore p : m = n : x$. (§ 252.)

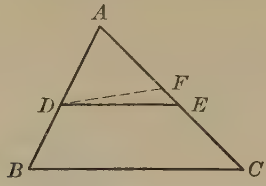
Ex. 28. Construct x so that $xn = m^2$.

Ex. 29. Construct x so that $2\,mn = px$.

Ex. 30. Let OB be any line within $\angle AOC$ and Y and Y' any two points on OB. Let YX and $Y'X'$ be perpendiculars to OA, and YZ and $Y'Z'$ be perpendiculars to OC. Prove that $OX : OX' = OZ : OZ'$.

* PROPOSITION IV. THEOREM

269. *A line which divides two sides of a triangle proportionally is parallel to the third side.*



Hypothesis. In $\triangle ABC$, DE intersects AB and AC so that

$$\frac{AB}{AD} = \frac{AC}{AE}.$$

Conclusion. $DE \parallel BC$.

Plan. Prove DE coincides with a line which is $\parallel$ BC.

Proof:

STATEMENTS	REASONS
1. Assume $DF \parallel BC$, meeting AC at F.	1. § 89.
2. $\therefore \dfrac{AB}{AD} = \dfrac{AC}{AF}.$	2. § 266.
3. But $\dfrac{AB}{AD} = \dfrac{AC}{AE}.$	3. Why?
4. $\therefore AF = AE.$	4. Why?
5. $\therefore$ F coincides with E, and DF with DE.	5. Why?
6. $\therefore DE \parallel BC.$	6. Since $DF \parallel BC$.

Ex. 31. Prove § 269, if $AD : DB = AE : EC$.

Plan. Prove BC coincides with $BX \parallel DE$, meeting AC at X.

Ex. 32. If $AD = 3$ in., $AB = 12$ in., $AC = 10$ in., and $AE = 2.5$ in., is $DE \parallel BC$?

Ex. 33. If $AD = 5$ in., $BD = 10$ in., $AE = 6$ in., and $EC = 11$ in., is $DE \parallel BC$?

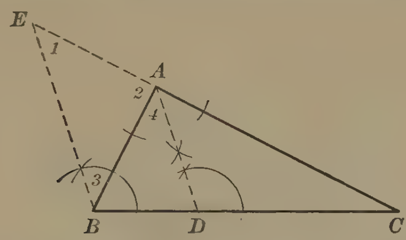
Ex. 34. What is the relation between Propositions I and IV?

270. If P is a point of AB, then P divides AB internally into two segments AP and PB.

Note. Additional Exercises 11–14, page 290, can be done now.

PROPOSITION V. THEOREM

271. *In any triangle, the bisector of an interior angle divides the opposite side internally into segments proportional to the adjacent sides of the triangle.*



Hypothesis. AD bisects $\angle A$ of $\triangle ABC$, meeting BC at D.

Conclusion. $\dfrac{BD}{DC} = \dfrac{BA}{AC}$.

Plan. Draw a parallel to DA, so as to find $BD : DC$.

Proof:

STATEMENTS	REASONS
1. Draw $BE \parallel DA$, meeting CA extended at E.	1. § 89.
2. Then, in $\triangle EBC$, $\dfrac{BD}{DC} = \dfrac{EA}{AC}$.	2. Why?

[The rest of the proof is left to the pupil.]

Suggestion. Prove $AB = AE$, and substitute in Step 2.

Ex. 35. In the figure above, find BD if $AB = 3$, $AC = 6$, and $DC = 5$.

Ex. 36. In the figure above, find AB if $BD = 2.5$, $AC = 4$, and $DC = 7.5$.

Ex. 37. The sides of a triangle are 10 in., 20 in., and 12 in. respectively. Find the segments of the side which is 12 in. long made by the bisector of the opposite angle.

Suggestion. Let x = the longer segment, and $12 - x$ the shorter.

Ex. 38. The sides of a triangle are 6 in., 7 in., and 8 in., respectively. Find the segments of each side made by the bisector of the opposite angle.

Note. Additional Exercises 15 to 20, p. 290, can be studied now.

SIMILAR POLYGONS

272. Introduction. The triangles below are similar triangles. Notice that they have the same shape.

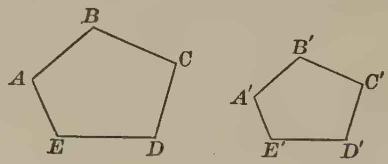
Ex. 39. Construct $\triangle ABC$ having $AB = 2$ in., $\angle A = 50°$, and $\angle B = 80°$. Also construct $\triangle A'B'C'$ having $A'B' = 4$ in., $\angle A' = 50°$, and $\angle B' = 80°$.

(*a*) Do the triangles appear to have the same shape?

(*b*) Find the lengths of BC, $B'C'$, AC, and $A'C'$.

(*c*) Find the values of the ratios $AB : A'B'$, $AC : A'C'$, and $BC : B'C'$. Are they approximately equal?

273. Two polygons are similar ($\sim$) if their corresponding angles are equal and their corresponding sides are proportional.



Thus, $ABCDE \sim A'B'C'D'E'$ if:

(1) $\angle A = \angle A'$; $\angle B = \angle B'$; $\angle C = \angle C'$; etc.; and

(2) $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'} = \text{etc.}$

274. The ratio of any two corresponding sides of two similar polygons is called **the ratio of similitude** of the polygons.

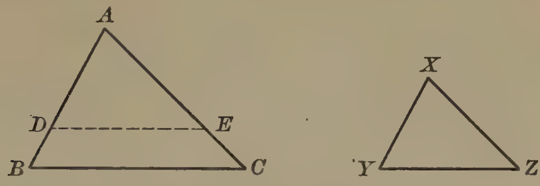
Ex. 40. (*a*) Are two squares necessarily similar? Why?

(*b*) Are two equilateral triangles? (*c*) Are two rectangles?

Ex. 41. The sides of one pentagon are 3, 4, 5, 8, and 11 in. respectively. The shortest side of a similar pentagon is 6 in. long. How long are the other sides of the second pentagon?

* PROPOSITION VI. THEOREM

275. *Two triangles are similar if they are mutually equiangular.*



Hypothesis. In $\triangle ABC$ and $\triangle XYZ$:

$$\angle A = \angle X;\quad \angle B = \angle Y;\quad \angle C = \angle Z.$$

Conclusion. $\triangle ABC \sim \triangle XYZ$.

Plan. Prove the corresponding sides proportional.

Proof:

STATEMENTS	REASONS
1. Place $\triangle XYZ$ in the position ADE, $\angle X$ coinciding with its equal $\angle A$.	1. § 60. § 22.
2. $\therefore \angle ADE = \angle Y = \angle ABC$.	2. Why?
3. $\therefore DE \parallel BC$.	3. Why?
4. $\therefore \dfrac{AB}{AD} = \dfrac{AC}{AE}$, or $\dfrac{AB}{XY} = \dfrac{AC}{XZ}$.	4. Why?
5. It can be proved that $\dfrac{AB}{XY} = \dfrac{BC}{YZ}$.	5. By a proof like that in Steps 1–4, placing $\triangle XYZ$ so $\angle Y$ coincides with $\angle B$.
6. $\therefore \dfrac{AB}{XY} = \dfrac{AC}{XZ} = \dfrac{BC}{YZ}$.	6. Why?
7. $\therefore \triangle ABC \sim \triangle XYZ$.	7. § 273.

276. Cor. 1. *Two triangles are similar if two angles of one are equal respectively to two angles of the other.*

277. Cor. 2. *Two right triangles are similar if an acute angle of one is equal to an acute angle of the other.*

Ex. 42. Prove that a line parallel to the base of a triangle, intersecting the other two sides, forms a triangle similar to the given triangle.

Ex. 43. Prove that any two equilateral triangles are similar.

Ex. 44. Prove that two isosceles triangles are similar if a base angle of one equals a base angle of the other.

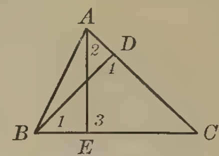
Ex. 45. Construct any $\triangle ABC$. Upon a segment XY which equals $2\,AB$, construct a triangle similar to $\triangle ABC$.

278. Corresponding sides of similar triangles are proportional, by the definition in § 273.

279. Fundamental Plan V. To prove four segments proportional, prove them corresponding sides of similar triangles.

Note. Select one triangle which has two of the segments as sides; select a second triangle which has the other two segments as sides. Prove the triangles similar. Select the corresponding sides.

A device for applying Fundamental Plan V. (Be sure to read Note 1 below.)



Hyp. In $\triangle ABC$: $AE \perp BC$;
$BD \perp AC$.

Con. $\dfrac{AC}{BC} = \dfrac{AE}{BD} = \dfrac{EC}{DC}.$

Plan. Prove the segments corresponding sides of similar triangles.

Proof:

STATEMENTS	REASONS
In $\triangle AEC$ and $\triangle BDC$:	
1. $AC \mid \angle 3 = \angle 4 \mid BC.$	1. Hypothesis.
2. $AE \mid \angle C = \angle C \mid BD.$	2. An $\angle$ equals itself.
3. $EC \mid \angle 2 = \angle 1 \mid DC.$	3. § 110.
4. $\therefore \triangle AEC \sim \triangle BDC.$	4. § 275.
5. $\therefore \dfrac{AC}{BC} = \dfrac{AE}{BD} = \dfrac{EC}{DC}.$	5. § 278.

Note 1. (*a*) Below $\triangle AEC$, write its three angles, and to the left of them write the sides which are opposite them in $\triangle AEC$.

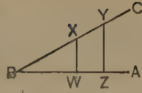
(*b*) Opposite the $\measuredangle$ of $\triangle AEC$ write the equal $\measuredangle$ of $\triangle BDC$.

(*c*) Beside the $\measuredangle$ of $\triangle BDC$, write the sides opposite in $\triangle BDC$.

(*d*) Then AC and BC are corresponding sides, since corresponding sides lie opposite equal $\measuredangle$; also AE and BD; also EC and DC. The ratios of Step 5 are obtained from Steps 1, 2, 3.

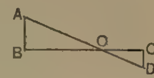
Note 2. If only two ratios are wanted, cross out the one not required by the conclusion.

Ex. 46. If X and Y are any two points on the side BC of acute $\angle ABC$, and XW and YZ are perpendiculars to AB, then: (*a*) $\triangle BXW \sim \triangle BYZ$; (*b*) $WX : YZ = BW : BZ$.



Ex. 47. If $AB \perp BC$ and $DC \perp BC$, prove that

$$\frac{AB}{DC} = \frac{BO}{CO} = \frac{AO}{DO}.$$



Ex. 48. If the altitudes AD and CE of $\triangle ABC$ intersect at F, prove that $AF : CF = EF : DF$.

Ex. 49. In the figure for Ex. 48, prove that $AD : CE = AB : CB$.

Ex. 50. Prove that the diagonals of a trapezoid divide each other so that the corresponding segments are proportional.

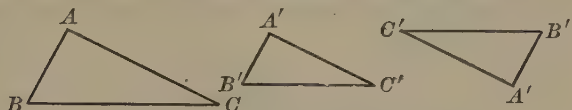
Ex. 51. If the chords AB and CD of a circle intersect at E, inside the circle, then $AE : DE = CE : EB$.

Ex. 52. If the altitudes AD and BE of $\triangle ABC$ intersect at O, then $AO : AC = OE : DC$.

Ex. 53. If the altitudes AD and BE of $\triangle ABC$ intersect at F, prove that $FD : EC = BD : BE$.

Ex. 54. In $\triangle ABC$, $\angle A = 50°$; $\angle B = 60°$; $AB = 3$ in.; and $AC = 2.75$ in. In $\triangle XYZ$, $\angle X = 50°$; $XZ = 5.5$ in.; $\angle Y = 60°$; and $YZ = 4\frac{7}{8}$ in. Find the remaining parts of the two triangles?

Ex. 55. Prove that two triangles are similar if their sides are parallel each to each. (Often given as a main theorem.)

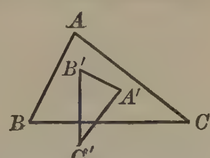


Hyp. In $\triangle ABC$ and (either) $\triangle A'B'C'$:
$AB \parallel A'B'$; $BC \parallel B'C'$; $AC \parallel A'C'$.

Con. $\triangle ABC \sim \triangle A'B'C'$.

Suggestion. Recall § 103.

Ex. 56. Prove that two triangles are similar if their sides are perpendicular each to each.



Hyp. In $\triangle ABC$ and $\triangle A'B'C'$:
$AB \perp A'B'$; $BC \perp B'C'$; $AC \perp A'C'$.

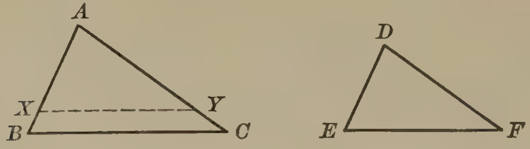
Con. $\triangle ABC \sim \triangle A'B'C'$.

Suggestion. Recall § 112.

Note. Additional Exercises 21 to 27, page 290, can be studied now.

* PROPOSITION VII. THEOREM

280. *Two triangles are similar if an angle of one equals an angle of the other and the sides including these angles are proportional.*



Hypothesis. In $\triangle ABC$ and $\triangle DEF$:

$$\angle A = \angle D;\ \frac{AB}{DE} = \frac{AC}{DF}.$$

Conclusion. $\triangle ABC \sim \triangle DEF$.

Plan. Prove two $\measuredangle$ of one $\triangle$ = to two $\measuredangle$ of the other $\triangle$.

Proof:

STATEMENTS	REASONS
1. Place $\triangle DEF$ on $\triangle ABC$, $\angle D$ coinciding with its equal, $\angle A$, E falling at X, and F falling at Y.	**1.** § 62. Why possible?
2. $\therefore \dfrac{AB}{AX} = \dfrac{AC}{AY}$.	**2.** Since $\dfrac{AB}{DE} = \dfrac{AC}{DF}$, by hyp.
3. $\therefore XY \parallel BC$.	**3.** Why?
4. $\therefore \triangle AXY \sim \triangle ABC$.	**4.** Give the proof.
5. $\therefore \triangle DEF \sim \triangle ABC$.	**5.** Since $\triangle AXY$ is $\triangle DEF$.

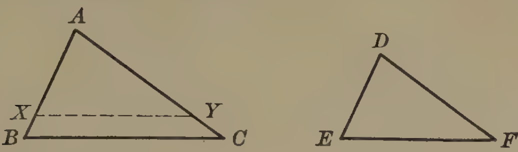
Ex. 57. On side AB of $\triangle ABC$, point X is located so that $AX = \frac{1}{3} AB$; on side AC, point Y is located so that $AY = \frac{1}{3} AC$. Prove $\triangle ABC \sim \triangle AXY$.

Ex. 58. On a straight line XY, lay off segments AB and DE so that $DE = 2\,AB$. Above XY, construct $BC \perp XY$ and $EF \perp XY$, so that $EF = 2\,BC$. Prove $DF = 2\,AC$, and $\angle A = \angle D$.

Ex. 59. The shadow of a chimney is 36 yd. long. At the same time the shadow of a stake 2 yd. high is 1.5 yd. long. How high is the chimney?

* PROPOSITION VIII. THEOREM

281. *Two triangles are similar if their corresponding sides are proportional.*



Hypothesis. In $\triangle ABC$ and $\triangle DEF$:

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}.$$

Conclusion. $\triangle ABC \sim \triangle DEF$.

Plan. On $\triangle ABC$, construct a triangle which can be proved similar to $\triangle ABC$ and congruent to $\triangle DEF$.

Proof:

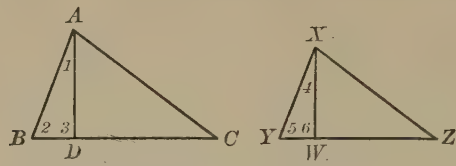
STATEMENTS	REASONS
1. On AB, take $AX = DE$. On AC take $AY = DF$. Draw XY, forming $\triangle AXY$.	1. These are all possible constructions.
2. $\therefore \frac{AB}{AX} = \frac{AC}{AY}$.	2. Since $\frac{AB}{DE} = \frac{AC}{DF}$, by hypothesis.
3. $\therefore XY \parallel BC$.	3. Why?
4. $\therefore \triangle AXY \sim \triangle ABC$.	4. Give the proof.
5. $\therefore \frac{AB}{AX} = \frac{BC}{XY}$, or $\frac{AB}{DE} = \frac{BC}{XY}$.	5. § 278; Ax. 2, § 49.
6. But $\frac{AB}{DE} = \frac{BC}{EF}$.	6. Why?
7. $\therefore XY = EF$.	7. § 254.
8. $\therefore \triangle DEF \cong \triangle AXY$.	8. Give the proof.
9. $\therefore \triangle DEF \sim \triangle ABC$.	9. Ax. 2, § 49.

Ex. 60. Construct any scalene triangle. Then construct a triangle whose sides are double the corresponding sides of the first triangle. Are the two triangles similar? How do their angles compare?

Ex. 61. Two segments AOB and COD intersect so that $AO = 3\,OB$, and $CO = 3\,OD$. Prove $\triangle AOC \sim \triangle BOD$, and $AC = 3\,BD$.

PROPOSITION IX. THEOREM

282. *Corresponding altitudes of similar triangles have the same ratio as any two corresponding sides.*



Hypothesis. $\triangle ABC \sim \triangle XYZ$.
AD and XW are corresponding altitudes.

Conclusion. $\dfrac{AD}{XW} = \dfrac{AB}{XY} = \dfrac{BC}{YZ} = \dfrac{AC}{XZ}$.

Plan. Prove AD, XW, AB, and XY corres. parts of similar $\triangle$s.

Proof:

STATEMENTS	REASONS
1. $\triangle ABD \sim \triangle XYW$.	1. Give the full proof.
2. $\therefore \dfrac{AD}{XW} = \dfrac{AB}{XY}$.	2. Why?
3. But $\dfrac{AB}{XY} = \dfrac{BC}{YZ} = \dfrac{AC}{XZ}$.	3. Why?
4. $\therefore \dfrac{AD}{XW} = \dfrac{AB}{XY} = \dfrac{BC}{YZ} = \dfrac{AC}{XZ}$.	4. Ax. 1, § 49.

Ex. 62. The base and altitude of a triangle are 5 ft. and 3 ft. respectively. If the corresponding base of a similar triangle is 7 ft., find its corresponding altitude.

Ex. 63. Prove that the bisectors of corresponding angles of similar triangles have the same ratio as any two corresponding sides.

Ex. 64. Prove that two corresponding medians of two similar triangles have the same ratio as any two corresponding sides of the triangles.

Suggestion. Use § 280.

Ex. 65. Determine three segments which shall bear to the sides of a given triangle the ratio 3 : 2. Then construct the triangle, having the new segments as sides. Are the two triangles similar?

Ex. 66. Prove that two isosceles triangles are similar if the vertex angle of one equals the vertex angle of the other.

Ex. 67. XA and ZB, the altitudes drawn from X and Z of $\triangle XYZ$, meet at O. Prove that $OA : BY = AZ : BZ$.

Ex. 68. RX and SY are the altitudes to the sides ST and RT of $\triangle RST$. If RX and SY intersect at Z, then $\triangle SXZ$ is similar to $\triangle RXT$.

Ex. 69. Altitudes AD and BE of $\triangle ABC$ intersect at F. Prove that $BF : AC = DF : DC$.

Ex. 70. $\angle A$ of $\triangle ABC$ is a right angle. From E, any point of AC, ED is drawn perpendicular to BC, meeting it at D.

(*a*) Examine the figure to discover a pair of similar triangles;

(*b*) prove the triangles similar;

(*c*) from these triangles determine the three equal ratios of sides of the triangle.

Ex. 71. $\angle ABC$ is an acute angle. CD is perpendicular to AB and AF is perpendicular to BC.

(*a*) Discover a pair of similar triangles;

(*b*) prove the triangles similar;

(*c*) write down three equal ratios of sides of these triangles.

Ex. 72. $\triangle AXY$ is inscribed in a semicircle having AY as diameter. From any point Z on XY extended through Y, a perpen-pendicular is drawn to meet AY extended at W.

Prove that $XA : WZ = YA : YZ$.

Ex. 73. If $\triangle ABC$ is a right triangle of which $\angle B = 90°$; if P is any point on AB and Q is any point on BC; if PR and QS are perpendiculars to AC from P and Q; then $AP : QC = AR : QS$.

Ex. 74. YW is the altitude to the hypotenuse XZ of right triangle XYZ. Prove that $YZ : ZW = ZX : YZ$.

Ex. 75. $\triangle AXY$ is inscribed in a semicircle having AY as diameter. From point Z on YX, ZW is drawn perpendicular to AY, meeting it at W. Prove that $ZW : XA = ZY : YA$.

Ex. 76. $\triangle ABC$ is inscribed in a circle. BD, the bisector of $\angle B$, meets AC at D, and the circle at E.

(*a*) Prove $\triangle ABE \sim \triangle BDC$.

(*b*) Prove $\triangle ABD \sim \triangle BEC$.

Ex. 77. In $\triangle ABC$, $AB = 3$ in.; $BC = 4$ in.; $AC = 6$ in. In $\triangle XYZ$, $XY = 12$ in.; $XZ = 6$ in.; $YZ = 8$ in.

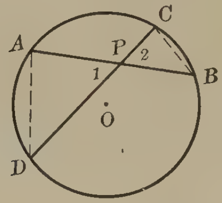
(*a*) Are the triangles similar? Why?

(*b*) If $\angle A = m°$; $\angle B = n°$; $\angle C = r°$; find $\measuredangle X$, Y, and Z.

Note. Additional Exercises 28 to 31, page 291, can be studied now.

* PROPOSITION X. THEOREM

283. *If two chords are drawn through a fixed point within a circle, the product of the segments of one is equal to the product of the segments of the other.*



Hypothesis. AB and CD are any two chords of $\odot\, O$ intersecting at point P.

Conclusion. $AP \cdot PB = DP \cdot PC.$

Analysis. If $AP \cdot PB = DP \cdot PC$, then $AP : PC = DP : PB$. (§ 252.) (Read § 286 at this time.)

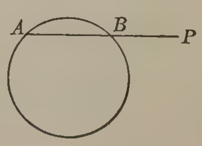
Plan. Use Fundamental Plan V. (§ 278.)

Proof:

STATEMENTS	REASONS
1. Draw AD and BC.	1. Why possible?
2. $\triangle APD \sim \triangle PBC$.	2. Give the full proof.
3. $\therefore \dfrac{AP}{PC} = \dfrac{DP}{PB}$.	3. Why?
4. $\therefore AP \cdot PB = PC \cdot DP$.	4. § 250.

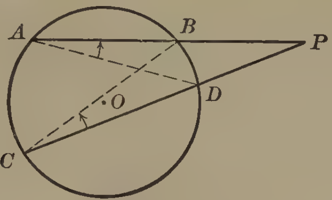
Ex. 78. Prove that the product of the segments of one diagonal of an inscribed quadrilateral is equal to the product of the segments of the other diagonal.

284. If a secant PA is drawn to a circle from a point P, cutting the circle at point B, then PA is called the **whole secant**, PB the **external segment**, and AB the **internal segment**.



PROPOSITION XI. THEOREM

285. *If any two secants are drawn through a fixed point outside a circle, the product of one and its external segment equals the product of the other and its external segment.*



Hypothesis. ABP and CDP are two secants of $\odot O$.

Conclusion. $AP \cdot BP = CP \cdot DP$.

Suggestion. Make an analysis, plan, and proof similar to that for Proposition X. Read § 286 at this time.

Ex. **79.** If the segments of AB in § 283 are 5 in. and 4 in. respectively, and $DP = 6$ in., how long is PC?

Ex. **80.** Chord CD bisects chord AB. The segments of CD are 3 in. and 12 in. long. How long is AB?

Ex. **81.** From a point P, a secant 18 in. long is drawn to a circle; the external segment is 4 in. The external segment of a second secant from the same point is 6 in. long. How long is the whole secant?

Ex. **82.** A secant is drawn from point P to a circle. The external segment is 4 in. and the internal segment is 11 in. How long must a second secant be in order that its internal segment shall be 3 in.?

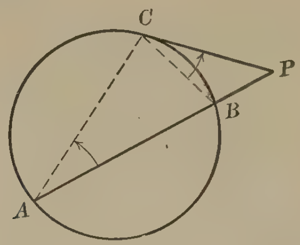
Ex. **83.** If any number of secants be drawn from an exterior point P, the product of the whole secant and the external segment is constant. (The conclusion means that the product of the whole secant and the external segment is the same for each secant.)

286. Fundamental Plan VI. To prove that the product of two segments equals the product of two other segments, first derive a proportion from the equation, using § 252; then prove the four segments corresponding sides of similar triangles; then write the product of the extremes equal to the product of the extremes.

Note. Additional Exercises 32 to 42, page 291, can be studied now.

PROPOSITION XII. THEOREM

287. *If a secant and a tangent are drawn to a circle from the same outside point, the square of the tangent equals the product of the whole secant and its external segment.*



Hypothesis. PC is tangent to $\odot O$; secant PA intersects the circle at B and A.

Conclusion. $\overline{PC}^2 = PA \cdot PB.$

Suggestion. Make an analysis, plan, and proof like that for Proposition X. (Read § 286, again.)

Note. Proposition XII may be stated: *If a secant and a tangent are drawn to a circle from the same point outside the circle, the tangent is the mean proportional between the whole secant and its external segment.*
For, when $\overline{PC}^2 = PA \cdot PB$, then $PA : PC = PC : PB$. Why?

Ex. 84. The length of the tangent to a circle from a point outside is 14 in. What must be the length of a secant from the same point in order that the external segment will be 7 in.?

Ex. 85. The tangent to a circle from an external point is 16 in. long, and the whole secant is 20 in. long. What is the length of the external segment of the secant?

Ex. 86. The shortest distance from a point to a circle of diameter 15 in. is 5 in. How long is the tangent from the same point?

Ex. 87. Repeat Ex. 86, when the diameter is 20 in., obtaining the result correct to one decimal place.

Ex. 88. The tangent to a circle from an external point is 15 in. long. How long must a secant from the same point be, in order that its external segment shall be 5 in. long?

Note. Additional Exercises 43 to 46, pg. 292, can be studied now.

Practice in Using Fundamental Plan VI

Ex. 89. If altitudes AD and BE of $\triangle ABC$ intersect at F, prove that the product of the segments of one is equal to the product of the segments of the other.

Ex. 90. If $\angle A$ of $\triangle ABC$ is a right angle and ED is drawn perpendicular to CB from any point E of AB, meeting CB at D, then prove that $EB \cdot AB = CB \cdot DB$.

Ex. 91. Prove that the product of one altitude of a triangle and its base equals the product of either other altitude and its base.

Ex. 92. If altitudes RX and SY of $\triangle RST$ intersect at O, prove that $RY \cdot TX = OY \cdot RX$.

Ex. 93. If altitudes AR and CS of $\triangle ABC$ intersect at O, prove that $AO \cdot CS = AS \cdot CB$.

Ex. 94. Let AC be the hypotenuse of right $\triangle ABC$. Let AD be perpendicular to AC and meet CB extended at D. Let CE be perpendicular to AC and meet AB extended at E.

Prove $\overline{AC}^2 = AD \cdot CE$.

Ex. 95. Point M bisects major arc AB of a circle. Chord MY, drawn from M to any point Y of minor arc AB, intersects AB at X. Prove chord AM is the mean proportional between MX and MY.

Ex. 96. AC is a chord and AB is a diameter of a circle. From F, any point on AC, FE is drawn perpendicular to AB, meeting it at E. Prove $AC \cdot AF = AB \cdot AE$.

Ex. 97. $\triangle ABC$ is an isosceles triangle having $AB = CA$. Point X is located on AC so that $BX = BC$. Prove that BC is the mean proportional between AC and CX.

Ex. 98. $\triangle ABC$ is an isosceles triangle having $AB = AC$. BD is perpendicular to AC at D. Prove that $\overline{BC}^2 = CD \times 2\,AC$.

Suggestion. Extend CA, through A, its own length to E. Draw BE. Prove $\angle CBE$ is a right $\angle$. Prove BC is the mean proportional between CD and CE.

Ex. 99. $\triangle ABC$ is inscribed in a circle O, which has line MN tangent to it at point A. From any point X of AB, a line XY is drawn parallel to MN, meeting AC at Y.

Prove that $AX \cdot AB = AY \cdot AC$.

Ex. 100. ⊙O and R are both tangent to line AB at point B. From A, a secant is drawn cutting $\odot O$ at D and C; and another is drawn cutting $\odot R$ at E and F. Prove $AC \cdot AD = AE \cdot AF$.

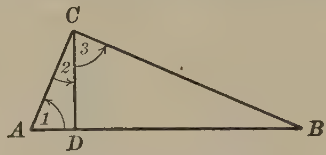
Suggestion. The ⊙ may be on the same side or on opposite sides of AB. Use § 287.

PROPOSITION XIII. THEOREM

288. *If the altitude be drawn to the hypotenuse of a right triangle:*

I. *The altitude is the mean proportional between the segments of the hypotenuse;*

II. *Each leg is the mean proportional between the whole hypotenuse and the adjacent segment.*



Hypothesis. In $\triangle ABC$, $\angle C$ is a rt. $\angle$.
$CD \perp AB$.

Conclusion I. $\dfrac{AD}{CD} = \dfrac{CD}{DB}$.

Plan. Prove the segments are corres. sides of similar $\triangle$s.

Proofs:

STATEMENTS	REASONS
1. $\angle 1$ is a comp. of $\angle 2$.	1. Why?
2. $\angle 3$ is a comp. of $\angle 2$.	2. Why?
3. $\therefore \angle 1 = \angle 3$.	3. Why?
4. $\triangle ADC \sim \triangle CDB$.	4. Give the full proof.
5. $\therefore AD : CD = CD : DB$.	5. Why?

Conclusion II. (*a*) $\dfrac{AD}{AC} = \dfrac{AC}{AB}$; (*b*) $\dfrac{DB}{CB} = \dfrac{CB}{AB}$.

Plan. Prove: (*a*) $\triangle ACD \sim \triangle ABC$; (*b*) $\triangle BCD \sim \triangle ABC$.

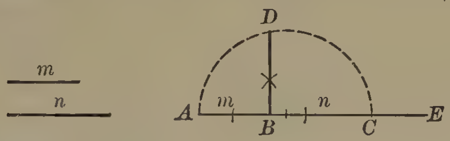
Ex. **101.** Find the altitude drawn to the hypotenuse of a right triangle if it divides the hypotenuse into two segments whose lengths are 3 in. and 12 in. respectively. Find each leg of the right triangle.

Ex. **102.** The hypotenuse of a right triangle is 20 in. and the perpendicular to it from the opposite vertex is 8 in. Find: (*a*) the segments of the hypotenuse; (*b*) find the two legs of the triangle.

Suggestion. Represent one segment of the hypotenuse by x. You will get a quadratic equation. See page 314.

PROPOSITION XIV. PROBLEM.

289. *Construct the mean proportional between two given segments.*



Given segments m and n.

Required to construct the mean proportional between m and n.

Construction. 1. On line AE, take $AB = m$ and $BC = n$.

2. Construct a semicircle on AC as diameter.

3. At B, construct $BD \perp AC$, meeting the semicircle at D.

Statement. BD is the mean proportional between m and n.
[Proof to be given by the pupil. Draw AD and CD. Use § 288.]

290. Cor. *If a perpendicular be drawn to the diameter of a circle from any point on the circle : (a) it is the mean proportional between the segments of the diameter;* and *(b) the chord drawn from the point to one end of the diameter is the mean proportional between the whole diameter and that segment of it which is adjacent to the chord.*

Ex. 103. If the diameter of a circle is 15 in., find the length of the chord which is perpendicular to the diameter at a point which is $4\frac{1}{2}$ in. from the center of the circle.

Ex. 104. Construct the mean proportional between 1 in. and 2 in.

Ex. 105. Construct $a\sqrt{3}$, where a is any segment.

Analysis. 1. Let $x = a\sqrt{3}$. Then $x^2 = 3a^2$.

2. $\therefore a : x = x : 3a$, or x is the mean proportional between a and $3a$.

[Construction left to the pupil.]

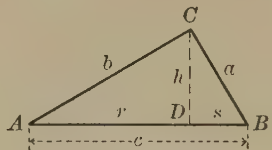
Ex. 106. If a and b are any segments, construct a segment which equals: (*a*) $\sqrt{3ab}$; (*b*) $\sqrt{\frac{2}{3}ab}$; (*c*) $\sqrt{(\frac{1}{3}a)(\frac{1}{2}b)}$; (*d*) $\sqrt{a(a+b)}$.

Ex. 107. In the figure for § 289, if $AB = 4$ in., and $BC = 12$ in., find the length of chord AD and of chord CD.

Note. Additional Exercises 47 to 54, page 292, can be studied now.

* PROPOSITION XV. THEOREM

291. *The square of the hypotenuse of a right triangle is equal to the sum of the squares of the legs.*



Hypothesis. In $\triangle ABC$, $\angle C$ is a right angle.

Conclusion. $c^2 = a^2 + b^2$.

Plan. Find a^2 and b^2 and add the results.

Proof:

	STATEMENTS	REASONS
1.	Draw $CD \perp AB$. Let $AD = r$, and $DB = s$.	1. § 82.
2.	$\therefore \frac{c}{a} = \frac{a}{s}$; and $\frac{c}{b} = \frac{b}{r}$.	2. § 288, II.
3.	$\therefore a^2 = cs$ and $b^2 = cr$.	3. Why?
4.	$\therefore a^2 + b^2 = cs + cr$.	4. Ax. 3, § 49.
5.	$\therefore a^2 + b^2 = c(s + r)$.	5. Factoring.
6.	$\therefore a^2 + b^2 = c \cdot c$, or $a^2 + b^2 = c^2$.	6. $c = s + r$.

Note. This theorem is called the Pythagorean Theorem, after Pythagoras, who formulated it. The theorem was evidently known even to the Egyptians. This proof of the theorem is attributed to Hindu mathematicians. In Book IV, we shall study Euclid's proof of the theorem — a strictly geometric proof, whereas this is more an algebraic one.

292. Cor. *The square of either leg of a right triangle equals the square of the hypotenuse minus the square of the other leg.*

Ex. 108. How long must a rope be to run from the top of a 12-foot tent pole to a point 16 ft. from the foot of the pole?

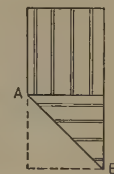
Ex. 109. The diameters of two concentric circles are 14 in. and 50 in., respectively. Find the length of a chord of the greater circle which is tangent to the smaller.

Ex. 110. A baseball diamond is a square whose sides are each 90 ft. long. What is the distance from "first" to "third"?

Ex. 111. How far is the pitcher of a base ball nine from each base when he is in the center of his box?

Ex. 112. The second base man caught a ball midway on the line between second and third bases. How far was he from first base?

Ex. 113. A piece of silk 27 in. wide is folded "on the bias" along line AB. How long is AB?



Ex. 114. What is the altitude of an isosceles triangle whose base is 16 in. long, and whose equal sides are each 12 in. long?

Ex. 115. How long is the altitude of an equilateral triangle whose sides are 12 in. long?

Ex. 116. How long is each side of the rhombus whose diagonals are 6 in. and 8 in. long respectively?

Ex. 117. The equal sides of an isosceles trapezoid are each 10 in. long. One of the bases is 30 in., and the other is 42 in. long. How long is the altitude of the trapezoid and the diagonal?

Ex. 118. The chords drawn from a point P on a circle to the ends of diameter AB are 6 in. and 8 in. long.

(*a*) How long is AB?
(*b*) How far from the center is each of the chords?

Ex. 119. If AD is the perpendicular from A to BC of $\triangle ABC$, prove

$$\overline{AB}^2 - \overline{AC}^2 = \overline{DB}^2 - \overline{CD}^2.$$

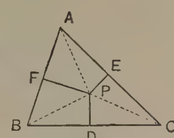
Plan. Find an expression for $\overline{AB}^2$ and $\overline{AC}^2$; then subtract the latter from the former. If the right member is not obtained at once, form it in the same manner, and try to prove the two results equal.

Ex. 120. If D is any point in the altitude from A to side BC of $\triangle ABC$, prove that $\overline{AB}^2 - \overline{AC}^2 = \overline{DB}^2 - \overline{DC}^2$.

Ex. 121. If a parallel to hypotenuse AB of right triangle ABC meets AC and BC at D and E respectively, prove that

$$\overline{AE}^2 + \overline{BD}^2 = \overline{AB}^2 + \overline{DE}^2.$$

Ex. 122. If perpendiculars PF, PD, and PE be drawn from any point P within an acute-angled triangle ABC to sides AB, BC, and CA respectively, prove that

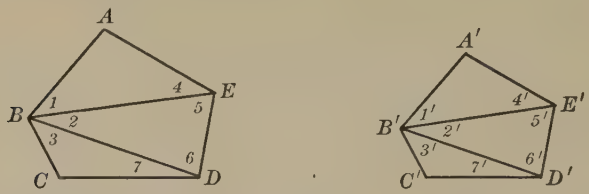


$$\overline{AF}^2 + \overline{BD}^2 + \overline{CE}^2 = \overline{AE}^2 + \overline{BF}^2 + \overline{CD}^2.$$

Note. Additional Exercises 55 to 79, p. 293, can be studied now.

PROPOSITION XVI. THEOREM

293. *Two polygons are similar if they are composed of the same number of triangles, similar each to each, and similarly placed.*



Hypothesis. $\triangle AEB \sim \triangle A'E'B'$; $\triangle EBD \sim \triangle E'B'D'$; $\triangle BCD \sim \triangle B'C'D'$.
The triangles are similarly placed.

Conclusion. Polygon $ABCDE \sim$ polygon $A'B'C'D'E'$.

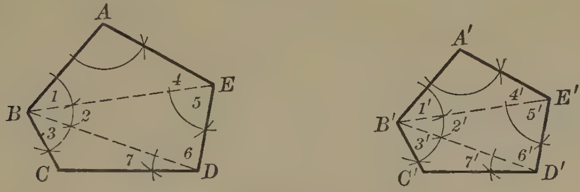
Plan. Prove corres. $\angle$s equal and corres. sides proportional.

Proof:

STATEMENTS	REASONS
1. $\angle 1 = \angle 1'$; $\angle 2 = \angle 2'$; and $\angle 3 = \angle 3'$.	**1.** Why?
2. $\therefore \angle B = \angle B'$.	**2.** Why?
3. Also, $\angle D = \angle D'$; and $\angle E = \angle E'$.	**3.** By similar proof.
4. $\angle A = \angle A'$; and $\angle C = \angle C'$.	**4.** Why?
5. $\dfrac{AB}{A'B'} = \dfrac{AE}{A'E'} = \dfrac{BE}{B'E'}$.	**5.** Why?
6. $\dfrac{BE}{B'E'} = \dfrac{ED}{E'D'} = \dfrac{BD}{B'D'}$.	**6.** Why?
7. $\dfrac{BD}{B'D'} = \dfrac{CD}{C'D'} = \dfrac{BC}{B'C'}$.	**7.** Why?
8. $\therefore \dfrac{AB}{A'B'} = \dfrac{AE}{A'E'} = \dfrac{ED}{E'D'} = \dfrac{CD}{C'D'} = \dfrac{BC}{B'C'}$.	**8.** Ax. 1, § 49.
9. $\therefore ABCDE \sim A'B'C'D'E'$.	**9.** § 273.

PROPOSITION XVII. PROBLEM

294. *Upon a given segment, corresponding to a given side of a given polygon, construct a polygon similar to the given polygon.*



Given polygon $ABCDE$ and segment $A'B'$.

Required to construct upon $A'B'$ as side corresponding to AB a polygon similar to $ABCDE$.

Plan. Base the construction on § 293.

Construction. 1. Divide $ABCDE$ into triangles by drawing EB and BD.

2. Construct $\triangle A'B'E'$ similar to $\triangle ABE$, by making $\angle A' = \angle A$, and $\angle 1' = \angle 1$.

3. Construct $\triangle E'B'D'$ similar to $\triangle EBD$. How?

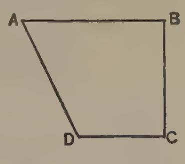
4. Construct $\triangle D'B'C'$ similar to $\triangle DBC$. How?

Statement. Polygon $A'B'C'D'E' \sim$ polygon $ABCDE$.

(Give the proof.)

Note. When studying this Proposition, review § 75. Then draw a pentagon, and "enlarge" it in the ratio of 2 : 1; that is, construct a pentagon similar to your selected pentagon, such that its segments will be twice as long as the corresponding segments of your given pentagon.

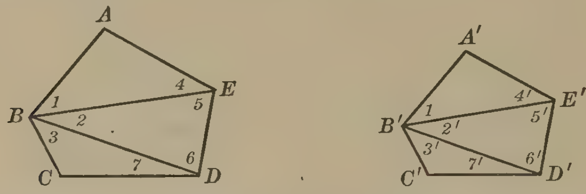
Ex. 123. $ABCD$ is the shape of an irregular piece of ground. Make a figure similar to $ABCD$ such that each side of the resulting figure shall be three times as long as the corresponding side of the given figure.



Ex. 124. Draw a rectangle. "Enlarge it in the ratio 3 to 1"; that is, construct a similar rectangle whose sides shall be 3 times as long as those of the given rectangle.

* PROPOSITION XVIII. THEOREM

295. *Two similar polygons can be separated into the same number of triangles, similar each to each, and similarly placed.*

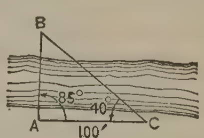


Hypothesis. Polygon $ABCDE \sim$ polygon $A'B'C'D'E'$, corresponding vertices being indicated by corresponding letters.

Conclusion. The polygons can be separated into the same number of $\triangle$, similar each to each and similarly placed.

Proof: STATEMENTS	REASONS
1. Draw BE, BD, $B'E'$, and $B'D'$.	1. Why possible?
2. $\angle A = \angle A'$.	2. Why?
3. $\frac{AB}{A'B'} = \frac{AE}{A'E'}$.	3. Why?
4. $\therefore \triangle ABE \sim \triangle A'B'E'$.	4. Why?
5. $\angle 4 = \angle 4'$; $\angle AED = \angle A'E'D'$.	5. Why?
6. $\therefore \angle 5 = \angle 5'$.	6. Ax. 4, § 49.
7. $\frac{BE}{B'E'} = \frac{AE}{A'E'}$; $\frac{ED}{E'D'} = \frac{AE}{A'E'}$.	7. Why?
8. $\therefore \frac{BE}{B'E'} = \frac{ED}{E'D'}$.	8. Why?
9. $\therefore \triangle BED \sim \triangle B'E'D'$.	9. Why?
10. Also $\triangle BDC \sim \triangle B'D'C'$.	10. By a similar proof.

Ex. 125. To find the distance AB across a stream, AC, $\angle A$, and $\angle C$ were measured and found to be as indicated in the adjoining figure. Construct a similar triangle, letting 1 in. represent 25 ft., and from the resulting figure find the length of AB.



296. Fundamental theorem about equal ratios.

In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

Hyp. $$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}.$$

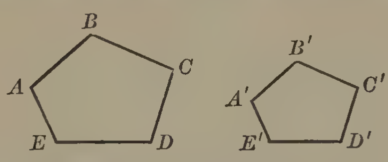
Con. $$\frac{a+c+e+g}{b+d+f+h} = \frac{a}{b} = \frac{c}{d} = \text{etc.}$$

Proof:

	Statements	Reasons
1.	Let $r = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}$.	1. Ax. 1, § 49.
2.	$\therefore br = a;\ dr = c;\ fr = e;\ hr = g.$	2. Give the proof.
3.	$\therefore br + dr + fr + hr = a + c + e + g.$	3. Ax. 3, § 49.
4.	$\therefore r(b + d + f + h) = a + c + e + g.$	4. Factoring.
5.	$\therefore r = \frac{a+c+e+g}{b+d+f+h}.$	5. Ax. 6, § 49.
6.	$\therefore \frac{a+c+e+g}{b+d+f+g} = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{b}.$	6. Ax. 2, § 49.

PROPOSITION XIX. THEOREM

297. *The perimeters of two similar polygons have the same ratio as any two corresponding sides.*



Hypothesis. $ABCDE \sim A'B'C'D'E'$, with corresponding vertices indicated by corresponding letters.

Conclusion. $$\frac{AB + BC + CD + DE + EA}{A'B' + B'C' + C'D' + D'E' + E'A'} = \frac{AB}{A'B'}, \text{ etc.}$$

Suggestions. 1. $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \text{etc.}$ Why?

2. Complete the proof. Apply § 296.

Note. Additional Exercises 80 to 85, p. 295, can be studied now.

298. Miscellaneous review. For a hasty review, have only the answers given; for a more searching review, have also the authorities given.

Ex. 126. Define the following terms:

(*a*) Ratio.
(*b*) Proportion.
(*c*) Antecedent.
(*d*) Consequent.
(*e*) Means.
(*f*) Extremes.
(*g*) Fourth proportional.
(*h*) Mean proportional.
(*i*) Similar polygons.
(*j*) Ratio of similitude.

Ex. 127. Write the proportion $\frac{x}{y} = \frac{z}{w}$:

(*a*) by inversion; (*b*) by alternation; (*c*) by addition.

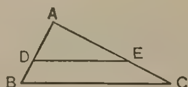
Ex. 128. If $\frac{x}{a} = \frac{y}{b}$, what do you know about ay?

Ex. 129. If $mn = sr$, write three proportions involving m, n, r, and s.

Ex. 130. (*a*) Are mutually equiangular triangles necessarily similar?
(*b*) Are mutually equiangular polygons necessarily similar?

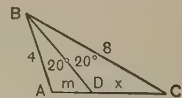
Ex. 131. If $\frac{a}{x} = \frac{a}{y}$, what do you know about x?

Ex. 132. If $\frac{a}{b} = \frac{c}{d}$, and $\frac{a}{b} = \frac{m}{d}$, what do you know about m?



Ex. 133. If $DE \parallel BC$, $AD = 8$, $DB = 2$, and $AE = 12$, how long is EC?

Ex. 134. If $AD = 6$, $AE = 9$, $DB = 2$, $EC = 3$, what do you know about DE?



Ex. 135. If $\angle ABD = 20°$, and $\angle DBC = 20°$, $AB = 4$, $BC = 8$, and $AD = m$, how long is DC?

Ex. 136. The base and altitude of one rectangle are 15 and 3 respectively; of a second rectangle, they are 20 and 5 respectively. Are the rectangles similar?

Ex. 137. In $\triangle ABC$: $\angle A = 40°$; $\angle B = 60°$; $AB = 6$ in. In $\triangle XYZ$: $\angle X = 40°$; $\angle Z = 60°$; $XZ = 12$ in. (Sketch the ▲.)
(*a*) What do you know about the triangles?
(*b*) If $AC = m$ in. and $BC = r$ in., how long is YZ?

Ex. 138. In rt. $\triangle ABC$, $\angle A = 90°$, $\angle C = 20°$. In $\triangle XYZ$: $\angle Y = 20°$; $\angle X = 90°$. (Sketch the ▲.)
(*a*) What do you know about the triangles?
(*b*) How long is XZ, if $XY = 24$ in., and $AB = \frac{1}{3} AC$?

Ex. 139. In $\triangle ABC$: $\angle A = 60°$; $AB = 24$ in.; $AC = 6$ in. In $\triangle XYZ$: $\angle Y = 60°$; $XY = 20$ in.; $YZ = 5$ in. (Sketch the $\triangle$s.)

(*a*) What do you know about the triangles?

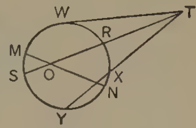
(*b*) How large are $\angle B$ and $\angle C$ if $\angle X = 50°$?

(*c*) How long is XZ if BC is m in.?

Ex. 140. In $\triangle ABC$: $AB = 2$; $AC = 3$; $BC = 4$. In $\triangle XYZ$: $XY = 6$; $YZ = 8$; $XZ = 4$. (Sketch the $\triangle$s.)

(*a*) What do you know about the triangles?

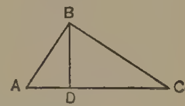
(*b*) What do you know about $\angle X$? $\angle B$? $\angle Y$?



Ex. 141. If $MO = 5$ in., $NO = 6$ in., and $RO = 3$ in., how long is SO?

Ex. 142. If $TS = 16$, $RT = 5$, $TX = 8$, find TY.

Ex. 143. If $TS = 32$, $RT = 2$, find tangent TW.



Ex. 144. If $\angle ABC = 90°$, $BD \perp AC$, $AD = 3$, $DC = 27$, find BD.

Ex. 145. If $\angle ABC = 90°$, $DB \perp AC$, $AD = 2$, $AC = 32$, find AB.

Ex. 146. If $\angle ABC = 90°$, $AB = 9$, $BC = 12$, find AC.

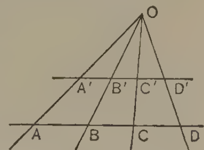
Ex. 147. If $\angle ABC = 90°$, $\overline{AB}^2 = 15$, $AC = 8$, find BC.

Ex. 148. Polygon P is similar to polygon R. The perimeter of P is 150. One side of P is 3, and the corresponding side of R is 9. What is the perimeter of R?

Ex. 149. The vertices of quadrilateral $ABCD$ are joined to a point O lying outside the quadrilateral. Points A', B', C', and D' are taken on OA, OB, OC, and OD, respectively, so that $A'B' \parallel AB$, $B'C' \parallel BC$, and $C'D' \parallel CD$. Prove $A'D' \parallel AD$.

Ex. 150. If P and S are two points on the same side of line OX such that the perpendiculars PR and ST drawn to OX have the same ratio as OR and OT, then points O, P, and S lie in a straight line.

Suggestion. Prove $\angle ROP = \angle TOS$ by proving $\triangle OPR \sim \triangle OST$.



Ex. 151. If two parallels are cut by three or more straight lines passing through a common point, the corresponding segments are proportional.

Ex. 152. Derive a formula for the altitude to the base of an isosceles triangle if the base is b and the equal sides are each a. By means of the formula determine the altitude when: (*a*) $a = 12$ and $b = 6$; (*b*) $a = 15$ and $b = 7$.

Note. Additional Exercises 86 to 99, page 296, can be studied now.

OPTIONAL TOPICS

Four optional topics follow, — all of which appear in some form in modern geometries. Excepting part of Topic C, these optional topics do not appear in the list of fundamental or subsidiary theorems of the College Entrance Requirements Board. Even when these topics cannot be studied by the class as a whole, some or all of the following material can be assigned to the more able pupils as special topics for which extra credit will be given.

Each group is independent of each of the others.

None of these theorems is required as an authority in the main lists of theorems of subsequent Books.

Topic A. **Scales and Scale Drawing.** (Page 185.)

This is probably the most common application of the fundamental ideas of Book III.

Topic B. **Trigonometric Ratios and Their Application.** (Page 187.)

A majority of experienced teachers favor teaching this subject at this point instead of in connection with ninth grade algebra. Not only is this the pedagogical place for it, but it is also the expedient place for it when preparing pupils for examinations of the College Entrance Requirements Board.

Topic C. **External and Harmonic Division of a Segment.** (Page 194.)

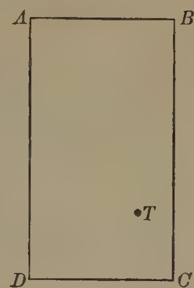
The main theorem of this topic appears in the list of fundamental theorems of the National Committee, and as a secondary theorem of the College Entrance Board. The balance of the topic has special mathematical interest.

Topic D. **Numerical Relations Among the Segments of a Triangle.** (Page 196.)

These theorems have appeared in geometries for a long time. They are among the best to exhibit correlation of arithmetic, algebra, and geometry.

Topic A. Scales and Scale Drawing

299. Scale drawings are a common and useful application of similar polygons.



The adjoining figure represents a lot $150' \times 275'$. It is drawn to the scale of $1''$ to $200'$; that is, AB, $\frac{3}{4}''$ in length, represents $150'$ and BC, $1\frac{3}{8}''$ in length, represents $275'$. If the corners of the lot itself are denoted by A', B', C', and D' respectively, then $AB : A'B' = 1 : 2400$, and $BC : B'C' = 1 : 2400$.

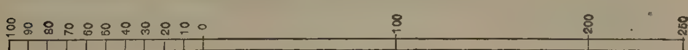
($1''$ to $200'$ is $1''$ to $2400''$ or $1 : 2400$.)

$ABCD$ is similar to $A'B'C'D'$, for the two figures are mutually equiangular (being rectangles) and their corresponding sides are proportional (the ratio of similitude being $1 : 2400$).

Since, in similar figures, the ratio of any two corresponding sides equals the ratio of similitude (§ 274), it is possible to determine from $ABCD$ the approximate length of any segment on the field itself.

300. Scales. The construction and use of scale drawings are made easy by the construction in advance of the scale itself.

Example. Below is part of the scale of $1''$ to $100'$.

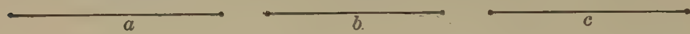


The segment extending from the zero mark to any division point represents the number of feet indicated above that point.

To determine the number of feet represented by a given segment according to the given scale: take the segment on the dividers; place one point of the dividers on a division mark at the right of the zero mark, so that the other point of the dividers will fall on the scale either at the zero mark or to the left of it. The length represented by the segment may then be read to the nearest 5 feet.

Thus, the segment below represents 235 ft. if drawn to the scale of $1''$ to $100'$.

Ex. 153. Determine the length represented by each of the following segments, assuming that they are drawn to the scale of 1″ to 100′.



Ex. 154. Determine the approximate number of feet represented by the diagonal AC in the figure of § 299.

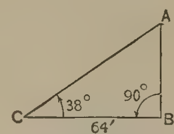
Note. You can use the scale on p. 185, even though the figure of § 299 is drawn to the scale 1″ : 200′. How?

Ex. 155. Determine the approximate distance of the tree, T, from each of the corners of the lot $ABCD$ in the figure of § 299.

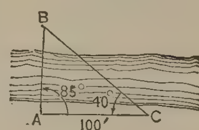
Ex. 156. Construct the scale of 1″ to 25′, to measure 100 ft., having the left-hand section show segments corresponding to 5′, 10′, etc., to 25′.

What length do segments a, b, c of Ex. 153 represent if it is assumed that they are drawn to the scale of 1″ to 25′?

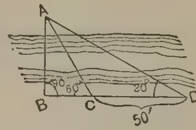
Ex. 157. Draw to the scale of 1″ to 25′ the adjoining figure. From the figure so drawn, determine the approximate length of AB.



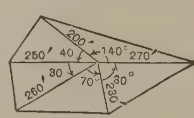
Ex. 158. Draw to the scale of 1″ to 25′ a figure similar to the adjoining one. From the resulting figure, determine the approximate distance represented by AB.



Ex. 159. Draw to the scale of 1″ to 25′ a figure similar to the adjoining one. From the resulting figure, determine the approximate length of AB.



Ex. 160. Draw to the scale of 1″ to 100′ a figure similar to the adjoining one. From the resulting figure, determine the approximate perimeter of the field having the dimensions indicated.



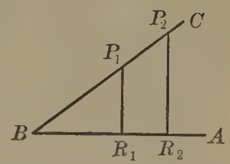
Ex. 161. (a) In a field $ABCD$, AB, BC, CD, and DA are 15, 25, 20, and 30 rods respectively, and diagonal AC is 35 rods. Make a scale drawing to the scale 1″ = 25 rd.

(b) From B and D draw the altitudes of $\triangle ABC$ and ADC. Measure them, — according to the scale 1″ = 25 rd.

(b) Using the rule "area of $\triangle = \frac{1}{2}$ base $\times$ altitude," find the area of $\triangle ABC$ and ACD, and of quadrilateral $ABCD$.

Topic B. Trigonometric Ratios

301. Sine of an angle. Let $\angle ABC$ be any angle. On BC, take any points P_1 and P_2. Draw perpendiculars P_1R_1 and P_2R_2 to AB.



Then $\triangle BP_1R_1 \sim \triangle BP_2R_2$. (Why?)

$$\therefore \frac{R_1P_1}{BP_1} = \frac{R_2P_2}{BP_2}.$$

That is, the ratio of the perpendicular RP to the distance BP is the same, wherever P_1 and P_2 are located on BC.

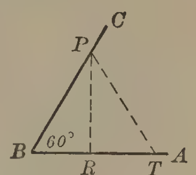
This constant ratio is called the **Sine of** $\angle B$. (**Sin** B.)

When the angle is acute, its sides and the perpendicular form a right triangle. In this triangle,

the sine of an acute angle = side opposite ÷ hypotenuse.

The sine of a given angle may be computed.

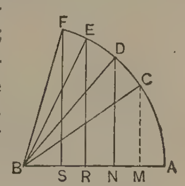
Example. Let $\angle B = 60°$. Determine sin 60°.



Solution. 1. Draw $PR \perp BA$. Draw $PT = PB$.
2. Then $\triangle PRT \cong \triangle PRB$, and $\angle T = 60°$.
3. $\therefore \triangle PBT$ is equilateral, and $BR = \frac{1}{2} BP$.
4. Let $BP = 2m$ and hence $BR = m$.
5. In $\triangle BPR$, $\overline{RP}^2 = 4m^2 - m^2$, or $3m^2$.
6. $\therefore RP = m\sqrt{3}$.
7. $\therefore \sin 60° = \frac{RP}{BP} = \frac{m\sqrt{3}}{2m} = \frac{1.732}{2} = .866^+$.

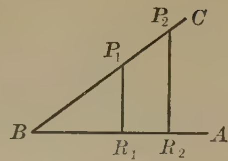
Ex. 162. Determine as above the value of sin 45° and of sin 30°.

Ex. 163. Construct a figure similar to the adjoining one making: $\angle ABC = 35°$; $\angle ABD = 50°$; $\angle ABE = 65°$; and $\angle ABF = 75°$. Draw the perpendiculars from C, D, E, and F to AB. Measure these perpendiculars and also the radius. Then compute the approximate values of the sine of each of the angles indicated; that is, of sin 35°, sin 50°, sin 65°, and sin 75°.



(If you have a metric scale, make AB = 100 mm. and measure the perpendiculars in mm.; if you do not have a metric scale, make $AB = 3\frac{1}{8}$ in., and measure the perpendiculars in sixteenths of an inch. Keep your figure for use in a later exercise.)

302. Cosine and tangent of an angle. It is easily proved, as in § 301, that the ratio $\frac{BR}{BP}$ is constant for all positions of P on BC; and also that $\frac{RP}{BR}$ is constant.



$\frac{BR}{BP}$ is called the **Cosine of** $\angle$**B.** (**Cos B.**)

$\frac{RP}{BR}$ is called the **Tangent of** $\angle$**B.** (**Tan B.**)

In the right triangle formed when $\angle B$ is an acute angle:

cosine of acute angle = adjacent side ÷ hypotenuse;

tangent of acute angle = opposite side ÷ adjacent side.

Example. When $\angle B = 60°$, if $BP = 2\,m$, then $BR = m$ and $RP = m\sqrt{3}$. (See Example, § 301.)

Hence,
$$\text{Cos } 60° = \frac{m}{2\,m} = \frac{1}{2} = .500.$$

$$\text{Tan } 60° = \frac{m\sqrt{3}}{m} = \sqrt{3} = 1.732.$$

Ex. 164. Compute the cosine and the tangent of 45° and 30° respectively.

303. Table of values of trigonometric ratios. The values of the sine, cosine, and tangent of the angles from 1° to 89° inclusive, to three decimal places, are given in the table on the opposite page. On page 317, the values correct to four places are given; these may be used if desired.

To determine the sine of 37° from the table: in the first column find 37°; on the same line with it, and in the column headed by the word Sin. is found .602. This is the sine of 37°.

The cosine of 37° is opposite 37° in the column headed Cos. It is .799.

The tangent of 37° is opposite 37° in the column headed Tan. It is .754.

TABLE OF VALUES OF TRIGONOMETRIC RATIOS

Angle	Sin	Cos	Tan	Angle	Sin	Cos	Tan
1°	.018	.999	.018	45°	.707	.707	1.000
2°	.035	.999	.035	46°	.719	.695	1.036
3°	.052	.998	.052	47°	.731	.682	1.072
4°	.070	.997	.070	48°	.743	.669	1.111
5°	.087	.996	.087	49°	.755	.656	1.150
6°	.105	.994	.105	50°	.766	.643	1.192
7°	.122	.992	.123	51°	.777	.629	1.235
8°	.139	.990	.141	52°	.788	.616	1.280
9°	.156	.988	.158	53°	.799	.602	1.327
10°	.174	.985	.176	54°	.809	.588	1.376
11°	.191	.982	.194	55°	.819	.574	1.428
12°	.208	.978	.213	56°	.829	.559	1.483
13°	.225	.974	.231	57°	.839	.545	1.540
14°	.242	.970	.249	58°	.848	.530	1.600
15°	.259	.966	.268	59°	.857	.515	1.664
16°	.276	.961	.287	60°	.866	.500	1.732
17°	.292	.956	.306	61°	.875	.485	1.804
18°	.309	.951	.325	62°	.883	.469	1.881
19°	.326	.946	.344	63°	.891	.454	1.963
20°	.342	.940	.364	64°	.899	.438	2.050
21°	.358	.934	.384	65°	.906	.423	2.144
22°	.375	.927	.404	66°	.914	.407	2.246
23°	.391	.921	.424	67°	.921	.391	2.356
24°	.407	.914	.445	68°	.927	.375	2.475
25°	.423	.906	.466	69°	.934	.358	2.605
26°	.438	.899	.488	70°	.940	.342	2.747
27°	.454	.891	.510	71°	.946	.326	2.904
28°	.469	.883	.532	72°	.951	.309	3.078
29°	.485	.875	.554	73°	.956	.292	3.271
30°	.500	.866	.577	74°	.961	.276	3.487
31°	.515	.857	.601	75°	.966	.259	3.732
32°	.530	.848	.625	76°	.970	.242	4.011
33°	.545	.839	.649	77°	.974	.225	4.331
34°	.559	.829	.675	78°	.978	.208	4.705
35°	.574	.819	.700	79°	.982	.191	5.145
36°	.588	.809	.727	80°	.985	.174	5.671
37°	.602	.799	.754	81°	.988	.156	6.314
38°	.616	.788	.781	82°	.990	.139	7.115
39°	.629	.777	.810	83°	.992	.122	8.144
40°	.643	.766	.839	84°	.994	.105	9.514
41°	.656	.755	.869	85°	.996	.087	11.430
42°	.669	.743	.900	86°	.997	.070	14.300
43°	.682	.731	.933	87°	.998	.052	19.081
44°	.695	.719	.966	88°	.999	.035	28.636
45°	.707	.707	1.000	89°	.999	.018	57.902

304. Practice in using the trigonometric table.

I. Observe: (*a*) the value of the sine increases as the angle increases.

(*b*) the value of the cosine decreases as the angle increases.

(*c*) the value of the tangent increases as the angle increases.

II. The *approximate value* of the sine, cosine, or tangent of *any angle* from 1° to 89° can be found by means of this table.

Example 1. Find sin 27° 25′.

Solution. 1.
$$\left.\begin{array}{ll}\text{Sin } 27^\circ & = .454 \\ \text{Sin } 27^\circ\, 25' & = \,? \\ \text{Sin } 28^\circ & = .469\end{array}\right\}\text{ difference } = .015$$

2. $25' = \frac{25}{60}$ or $\frac{5}{12}$ of 1°. $\frac{5}{12} \times .015 = .006.$

3. Since the sine increases as the angle increases,
$\text{Sin } 27^\circ\, 25' = .454 + .006$, or .460.

Example 2. Find cos 62° 15′.

Solution. 1.
$$\left.\begin{array}{ll}\text{Cos } 62^\circ & = .469 \\ \text{Cos } 62^\circ\, 15' & = \,? \\ \text{Cos } 63^\circ & = .454\end{array}\right\}\text{ difference } = .015.$$

2. $15' = \frac{1}{4}$ of 1°. $\frac{1}{4} \times .015 = .0037^{+}$, or .004.

3. Since the cosine decreases as the angle increases,
$\text{Cos } 62^\circ\, 15' = .469 - .004$, or .465.

Example 3. Find tangent 72° 35′.

Solution. 1.
$$\left.\begin{array}{ll}\text{Tan } 72^\circ & = 3.078 \\ \text{Tan } 72^\circ\, 35' & = \,? \\ \text{Tan } 73^\circ & = 3.271\end{array}\right\}\text{ difference } = .193$$

2. $35' = \frac{35}{60}$ or $\frac{7}{12}$ of 1°. $\frac{7}{12} \times .193 = .1125$, or .113.

3. Since the tangent increases as the angle increases,
$\text{Tan } 72^\circ\, 35' = 3.078 + .113$, or 3.191.

Example 4. What is the angle x when $\sin x = .635$?

Solution. 1.
$$\left.\begin{array}{l}\left.\begin{array}{ll}\text{Sin } 39^\circ & = .629 \\ \text{Sin } x & = .635\end{array}\right\}\begin{array}{l}\text{difference} \\ = .006\end{array} \\ \begin{array}{ll}\text{Sin } 40^\circ & = .643\end{array}\end{array}\right\}\begin{array}{l}\text{difference} \\ = .014\end{array}$$

2. $.006 \div .014 = .42^{+}$. $.42$ of $60' = 25.20'$ or $25'$

3. $\therefore$ angle $x = 39^\circ + 25'$ (about).

Ex. 165. Find: sin 43°; sin 37°; sin 65°.

Ex. 166. Find: cos 36°; cos 73°; cos 48°.

Ex. 167. Find: tan 24°; tan 78°; tan 29°.

Ex. 168. Find $\angle x$ if: $\sin x = .515$; $\sin x = .899$; $\sin x = .656$.

Ex. 169. Find $\angle x$ if: $\cos x = .996$; $\cos x = .743$; $\cos x = .156$.

Ex. 170. Find $\angle x$ if: $\tan x = .306$; $\tan x = .727$; $\tan x = 11.430$.

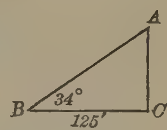
Ex. 171. Find: sin 30° 20′; sin 30° 40′; sin 30° 45′.

Ex. 172. Find: cos 43° 15′; cos 43° 10′; cos 43° 25′.

Ex. 173. Find: tan 52° 30′; tan 40° 15′; tan 25° 45′.

305. Finding lengths, distances, and angles by means of the trigonometric ratios.

Example 1. Assume that AC represents a pole of unknown height; that B is 125 ft. from its foot; that the imaginary line AB makes with BC an angle measuring 34°. Find AC.



Solution. 1. Let $AC = x$ feet. The ratio which uses the sides AC and BC is the tangent of angle B.

$$\text{Tan } B = \frac{AC}{BC}. \quad \therefore \tan 34° = \frac{x}{125}.$$

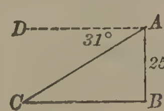
2. $\therefore x = 125 \times \tan 34°.$

3. $\therefore x = 125 \times .675$, or 84.375 ft., or about 84 ft.

Note 1. tan 34° = .675, from the table.

Note 2. Angle CBA is called the **angle of elevation** of point A at point B.

Example 2. AB represents a lighthouse 250 ft. high. AD is an imaginary line parallel to BC. C represents the position of a ship. $\angle DAC = 31°$. Find BC, the distance of the ship from a point immediately below the lighthouse.



Solution. 1. In such a figure, $\angle BCA$ equals $\angle DAC$. $\therefore \angle BCA = 31°$.

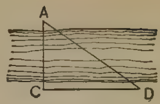
2. $$\text{Tan } BCA = \frac{AB}{BC}.$$

3. $\therefore BC \times \tan 31° = 250.$ (Tan 31° = .601 in the table.)

4. $\therefore BC \times .601 = 250.$

5. D$_{.601}$ $BC = 415.9$ ft.

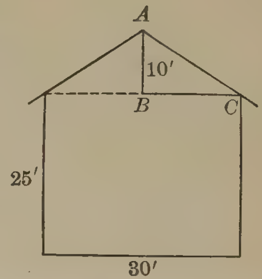
Note. Angle DAC is called the **angle of depression** of point C at point A.



Ex. 174. In the adjoining figure, AC is perpendicular to CD; $CD = 150$ ft.; angle $D = 62°\ 30'$. Find AC and AD.

Ex. 175. 165 feet from the foot of a high building the angle of elevation of the top is 35° 20′. How high is the building?

Ex. 176. From the top of a hill known to be 175 ft. above the level of the plain, the angle of depression of a house is 22° 45′. How far away is the house from an imaginary point directly below the top of the hill?



Ex. 177. In $\triangle ABC$, BC is 18 ft., $\angle B$ is 65° 20′, and AB is 8 ft. Draw a triangle to represent these conditions and draw the altitude AD.

a. Find the length of AD.
b. Having found AD, find the area.

Ex. 178. In the figure at the right is a right triangle in which $AB = 10$ ft. and $BC = 15'$.

a. Find $\angle C$.
b. Find AC.

Ex. 179. At a point 125 ft. from the foot of a high building, the angle of elevation of the top is 47° 30′. How high is the building?

Ex. 180. From a height of 250 ft., the angle of depression of an object on the plain below is 32° 15′. Find the distance of the object from a point in the plain directly below the point of observation.

Ex. 181. In $\triangle ABC$, in which $\angle C$ is a right angle, and $\angle B$ is an angle of 24° 45′, side AB is 22 in. long. Find the length of AC and of BC.

Ex. 182. The angle of elevation of an aëroplane at a certain point P is 48° 30′. Point D, 1500 ft. distant, is directly below the aëroplane. How high is the aëroplane?

Ex. 183. From a hilltop 175 feet above the level of a lake, the angle of depression of one sailboat is 42° 40′. The angle of depression of a second boat directly in line with the first boat is 68° 20′. What is the distance between the two boats?

Ex. 184. An observer in an aëroplane, which is 1800 feet high, finds that the angle of depression of a station on the ground is 25°. How far is it from his present position to a point directly over the station?

Ex. 185. What is the angle of elevation of the sun, if a rod 8 feet long throws a shadow 3 feet long?

Ex. 186. At a time when the angle of elevation of the sun is known to be $35° 15'$, a chimney casts a shadow 85 feet long. How high is the chimney?

Ex. 187. If the radius of a circle is 34 in., what is the length of a chord which subtends an arc of $43°$?

Ex. 188. If a circle of radius 5 in. be divided into five equal arcs, what is the length of the chord subtending one of the arcs?

Ex. 189. The mid-point of a chord 24 in. long is 7 in. from the mid-point of the arc subtended by the chord. How many degrees are there in the arc subtended by the chord?

Ex. 190. In rt. $\triangle ABC$, having $\angle A = 90°$, $\angle B = 35°$, and $AB = 15$ in., how long are: AC; BC; the altitude AD to BC; and BD?

Ex. 191. In $\triangle ABC$, having $AB = AC$, and altitude AD drawn to base BC, find $\angle A$ and side BC if $AB = 25$ in. and $AD = 18$ in.

Ex. 192. At a point 169 ft. from the foot of a tower, on which stands a flagpole, the angle of elevation of the top of the tower is $35°$, and of the top of the pole $47°$. How high is the tower, and how long is the pole?

Ex. 193. How long is the radius of the circle which is inscribed in an equilateral triangle whose sides are 12 in. long?

Ex. 194. From a point 200 feet from the foot of a tower surmounted by a flagpole, the angle of elevation of the top of the pole is $38°$; from a point 150 ft. further away in a straight line with the tower, the angle of elevation of the foot of the pole is $22°$. How long is the pole?

Ex. 195. Find the altitude of $\triangle ABC$ if the base $BC = 30$, the side $AB = 20$, and:

(*a*) $\angle A = 40°$; (*b*) $\angle A = 50°$; (*c*) $\angle A = 70°$.

(*d*) If BC and AB remain unchanged, what happens to the altitude as $\angle A$ increases?

Ex. 196. Find the altitude of the parallelogram $ABCD$, if its base $BC = 40$, its side $AB = 20$, and:

(*a*) $\angle A = 35°$; (*b*) $\angle A = 55°$; (*c*) $\angle A = 75°$.

(*d*) If AB and BC remain unchanged, what happens to the altitude as $\angle A$ increases?

Ex. 197. In right $\triangle ABC$, having $\angle A = 90°$, AD is perpendicular to BC. If $\angle B = 40°$, and $AB = 12$ in., find

(*a*) AD; (*b*) BD; (*c*) AC; (*d*) CD; (*e*) BC.

Optional Topic C

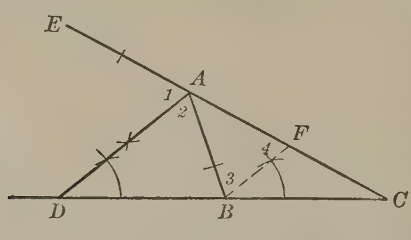
EXTERNAL AND HARMONIC DIVISION OF A SEGMENT

306. External division of a segment. If P is a point on line AB, but not between A and B, then P is said to divide AB externally into the segments AP and PB. The segments should be read from the beginning of the segment, A, to the division point, P; and from P to the other end of the given segment, B.

If P were an internal point of AB, $AP + PB = AB$.
If AB is considered positive, BA is considered negative, and $AB + BA = O$. Similarly $AP + PA = O$. With this understanding, $AP + PB = AB$, in both figures above, even when P is an external division point.

PROPOSITION XX. THEOREM

307. *In any triangle, the bisector of an exterior angle at any vertex divides the opposite side externally into two segments whose ratio equals the ratio of the two adjacent sides of the triangle.*



Hypothesis. AD bisects ext. $\angle BAE$ of $\triangle ABC$, meeting CB extended at D.

Conclusion. $BD : DC = BA : AC$.

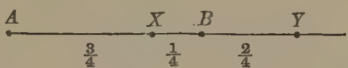
Plan. Draw a $\parallel$ to AD through B, and find a ratio equal to $BD : DC$.

(Proof on page 195.)

Proof: STATEMENTS	REASONS
1. Draw $BF \parallel DA$, meeting AC at F.	1. Why possible?
2. $\therefore BD:DC = FA:AC$.	2. Why?
3. $\angle 3 = \angle 2$; $\angle 4 = \angle 1$.	3. Give the proof.
4. $\therefore \angle 3 = \angle 4$.	4. Give the proof.
5. $\therefore BA = FA$.	5. Why?
6. $BD:DC = BA:AC$.	6. Why?

Ex. 193. The sides of a triangle are $AB = 6$, $AC = 9$, and $BC = 12$. Find the segments into which AC is divided by the bisector of the exterior angle at the opposite vertex.

308. A segment is **divided harmonically** if it is divided internally and externally into segments which have the same ratio.

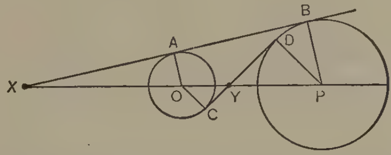


Thus, if $AB = 1$, $AX = \frac{3}{4}$, and $AY = \frac{3}{2}$, then $\frac{AX}{XB} = \frac{AY}{YB}$ since each of these ratios equals $\frac{3}{1}$. Hence X and Y divide AB harmonically.

Ex. 199. Prove that the bisector of the interior angle of a triangle and the bisector of the exterior angle at the same vertex divide the opposite side harmonically.

Suggestion. Apply § 271 and § 307.

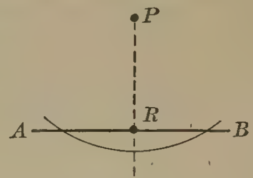
Ex. 200. Prove that the common internal tangent of two unequal circles divides the line of centers internally into two segments which have the same ratio as the radii of the circles.



Ex. 201. Prove that the common external tangent of two unequal circles divides the line of centers externally into two segments which have the same ratio as the radii of the circles.

Optional Topic D

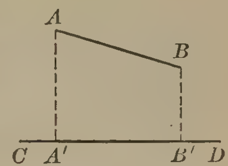
NUMERICAL RELATIONS AMONG SEGMENTS OF A TRIANGLE



309. The projection of a point upon a given line is the foot of the perpendicular drawn from the point to the line.

Thus, R is the projection of P on AB.

PR is only the means of obtaining R. It may be called the *projector* of P on AB.



310. The projection of a segment upon a given line is the distance between the projections of its end-points.

Thus, the projection of AB on line CD is $A'B'$.

The symbol "p_{CD}^{AB}" is read "the projection of AB on CD."

Ex. 202. Draw a segment AB and also four straight lines not parallel to AB but also not *crossing* AB.

(*a*) Determine the projection of AB on each of the straight lines.

(*b*) Are the projections all of the same length?

Ex. 203. Draw an acute scalene triangle. Show by means of a drawing the projection of the shortest side upon each of the other sides.

Ex. 204. Repeat the preceding exercise for an obtuse triangle.

Ex. 205. Draw an obtuse triangle. Obtain the projection of the longest side upon each of the other sides.

Ex. 206. If AB, extended, makes an acute angle with a line m, prove that p_m^{AB} is less than AB.

Ex. 207. If $AB \parallel m$, how does p_m^{AB} compare with segment AB?

Ex. 208. If $AB \perp m$, what is the length of p_m^{AB}?

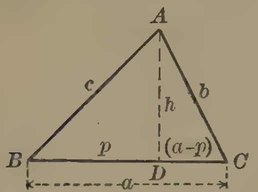
Ex. 209. What part of the base of an isosceles triangle is the projection upon the base of one of the equal sides?

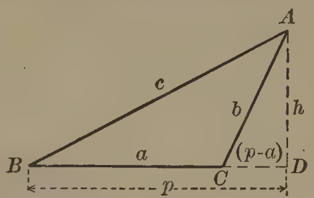
Ex. 210. If the equal sides of an isosceles trapezoid be projected upon the lower base, the projections are equal.

Ex. 211. Draw a right triangle and draw the median to the hypotenuse. Prove that the projection of the median upon either leg of the triangle is one half of that leg.

PROPOSITION XXI. THEOREM

311. *In any triangle, the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides, minus twice the product of one of these sides and the projection of the other upon it.*





Hypothesis. In $\triangle ABC$, $\angle B$ is acute.

Conclusion. $b^2 = a^2 + c^2 - 2\,a \cdot p_a^c$.

Proof:

STATEMENTS	REASONS
1. Draw $AD \perp BC$.	1. Why possible?
2. Let $p = BD$, or p_a^c. $\therefore DC = a - p$.	2. Ax. 7, § 49.
3. In Fig. 1, $b^2 = h^2 + (a - p)^2$.	3. Why?
4. But $h^2 = c^2 - p^2$.	4. Why?

(Complete the proof, by substituting for h^2 in Step 3.)

Note 1. A similar proof may be given from Fig. 2. $DC = p - a$.

Note 2. The conclusion of Proposition XXI is a formula connecting the three sides of a triangle with the projection of one side upon one of the other two sides. Altogether four different numbers are involved. Hence, when three of these numbers are known, the fourth may be determined by substituting in the formula and solving the resulting equation. In the right member, there appear the squares of two sides and the projection of one of these upon the other; in the left member, there appears the square of the third side.

Ex. 212. Write the formula for a^2; for c^2.

Ex. 213. Determine p_b^a when $a = 13$, $b = 14$, and $c = 15$.

Ex. 214. Determine b when $a = 10$, $c = 12$, and $p_a^c = 9$.

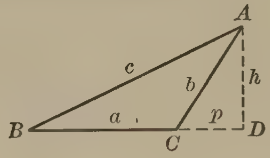
Ex. 215. Determine c when $a = 11$, $b = 16$, and $p_a^c = 7$.

Ex. 216. Determine a when $b = 18$, $c = 12$, and $p_a^c = 4$.

Note. Additional Exercises 100 to 108, page 297, can be studied now.

PROPOSITION XXII. THEOREM

312. *In any triangle having an obtuse angle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, plus twice the product of one of these sides and the projection of the other side upon it.*



Hypothesis. In $\triangle ABC$, $\angle C$ is an obtuse $\angle$.

Conclusion. $c^2 = a^2 + b^2 + 2\,a \cdot p_a^b$.

Proof:

STATEMENTS	REASONS
1. Draw $AD \perp BC$.	1. Why possible?
2. Let $p = CD$, or p_a^b. $\therefore BD = a + p$.	2. Ax. 7, § 49.
3. In $\triangle ABD$, $c^2 = h^2 + (a+p)^2$.	3. Why?
4. In $\triangle ACD$, $h^2 = b^2 - p^2$.	4. Why?

(Complete the proof by substituting for h^2 in Step 3.)

313. Cor. If a, b, and c are the sides of a triangle:

$\angle A$ is acute if $a^2 < b^2 + c^2$;
$\angle A$ is obtuse if $a^2 > b^2 + c^2$;
$\angle A$ is a right $\angle$ if $a^2 = b^2 + c^2$.
(Proof is indirect in each case.)

Ex. 217. Is the greatest angle of the triangle whose sides are 12, 35, and 37 acute, right, or obtuse?

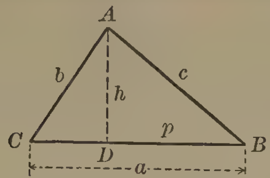
Ex. 218. Prove that the sum of the squares of the diagonals of a parallelogram equals the sum of the squares of the sides of the parallelogram. (Use § 311 and § 312.)

Ex. 219. If $c = 12$, $a = 7$, $b = 3$, find p_7^3.

Ex. 220. Solve the formula $c^2 = a^2 + b^2 + 2\,ap$, for p.

Ex. 221. Solve the formula $c^2 = a^2 + b^2 + 2\,ap$, for a.

314. Formula for the altitude of a triangle.



1. Assume $AD = h_a$, and $\angle B$ to be an acute angle.

2. $\therefore b^2 = a^2 + c^2 - 2a \cdot p_a^c$ or $b^2 = a^2 + c^2 - 2ap$.

3. $$\therefore p = \frac{a^2 + c^2 - b^2}{2a}.$$

4. $$h_a^2 = c^2 - p^2 = (c + p)(c - p).$$

5. $$\therefore h_a^2 = \left[c + \frac{a^2 + c^2 - b^2}{2a}\right]\left[c - \frac{a^2 + c^2 - b^2}{2a}\right]$$

6. $$= \left[\frac{2ac + a^2 + c^2 - b^2}{2a}\right]\left[\frac{2ac - a^2 - c^2 + b^2}{2a}\right]$$

7. $$= \frac{[(a + c)^2 - b^2][b^2 - (a - c)^2]}{4a^2}$$

8. $$= \frac{(a+c+b)(a+c-b)(b+a-c)(b-a+c)}{4a^2}.$$

9. Let $a + b + c = 2s$.

10. $\therefore a + b - c = 2s - 2c = 2(s - c)$.

Similarly, $b + c - a = 2(s - a)$; and $c + a - b = 2(s - b)$.

11. $$\therefore h_a^2 = \frac{2s \cdot 2(s - b) \cdot 2 \cdot (s - c) \cdot 2(s - a)}{4a^2}$$

$$= \frac{4s(s - a)(s - b)(s - c)}{a^2}.$$

12. $$\therefore h_a = \frac{2}{a}\sqrt{s(s - a)(s - b)(s - c)}.$$

Similarly, $$h_b = \frac{2}{b}\sqrt{s(s - a)(s - b)(s - c)};$$

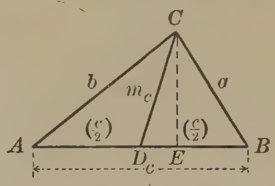
and $$h_c = \frac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}.$$

Ex. 222. Find the three altitudes of the triangle whose sides are 13, 14, and 15, getting the results correct to one decimal place.

Note. Additional Exercises 109 to 115, page 297, can be studied now.

PROPOSITION XXIII. THEOREM

315. *In any triangle, the sum of the squares of two sides equals twice the square of half the third side plus twice the square of the median drawn to that side.*



Hypothesis. In $\triangle ABC$, CD is the median to side AB.

Conclusion. $a^2 + b^2 = 2\left(\frac{c}{2}\right)^2 + 2\,m_c^2.$

Plan. 1. Assume $\angle ADC$ is obtuse and $\angle BDC$ acute.

2. Draw $CE \perp AB$, so that $DE = p_{AB}^{CD}$.

3. Find a^2 from $\triangle BCD$ by § 311; and b^2 from $\triangle ACD$, by § 312.

4. Add the results obtained in the last suggestion.

(Proof left to the pupil.)

Note 1. $\angle ADC$ is right, acute, or obtuse. If it is a right angle, the proof is quite easy.

Note 2. By Proposition XXIII, it is possible to determine the three medians of a triangle when the three sides of the triangle are known.

316. Cor. *The difference between the squares of two sides of a triangle equals twice the product of the third side and the projection of the median upon that side.*

Con. $b^2 - a^2 = 2\,c \cdot p_c^{m_c}.$

Suggestion. Determine b^2 and a^2 and then subtract the value of a^2 from that of b^2.

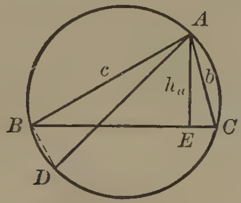
Ex. 223. Determine m_a when $b = 12$, $c = 16$, and $a = 20$. Determine also m_b and m_c.

Ex. 224. From the conclusion of § 315 derive a formula for m_c in terms of a, b, and c.

Ex. 225. Prove that the medians to the equal sides (s, s) of an isosceles $\triangle$ of base b are equal. (Use § 315.)

PROPOSITION XXIV. THEOREM

317. *In any triangle, the product of two sides equals the product of the diameter of the circumscribed circle and the altitude upon the third side.*



Hypothesis. $\triangle ABC$ is inscribed in $\odot O$; AD is a diameter of $\odot O$; $AE = h_a$.

Conclusion. $b \cdot c = d \cdot h_a$.

Analysis and proof left to the pupil. See analysis of § 284.

318. By § 314, $\quad h_a = \dfrac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$.

Hence, by § 317, $\quad b \cdot c = \dfrac{2\,d}{a}\sqrt{s(s-a)(s-b)(s-c)}$.

$$\therefore\ 2\,d\sqrt{s(s-a)(s-b)(s-c)} = abc,$$

or
$$d = \frac{abc}{2\sqrt{s(s-a)(s-b)(s-c)}}.$$

Hence, when the sides of a triangle are known, the diameter of the circumscribed circle can be computed.

Ex. 226. Determine the diameter of the circle circumscribed about the triangle whose sides are 13, 14, and 15.

Ex. 227. If the sides of $\triangle ABC$ are 10, 14, and 16, find the lengths of the altitude and the median drawn to the side 16. Determine also the diameter of the circumscribed circle.

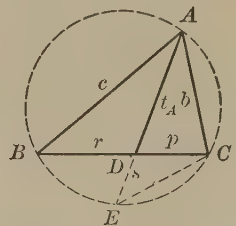
Ex. 228. In the equilateral $\triangle$ whose sides each equal s, find the length of the altitude and the median to one side.

Ex. 229. Prove that the altitudes to the equal sides (s, s) of an isosceles $\triangle$ (of base b) are equal. (Use § 314.)

Note. Additional Exercises 116 to 118, page 298, can be studied now.

PROPOSITION XXV. THEOREM

319. *In any triangle, the product of any two sides is equal to the product of the segments of the third side formed by the bisector of the opposite angle, plus the square of the bisector.*



Hypothesis. AD bisects $\angle A$ of $\triangle ABC$, meeting BC at D. (Let $BD = r$, and $DC = p$.)

Conclusion. $b \cdot c = t_A^2 + r \cdot p$.

Proof:

STATEMENTS	REASONS
1. Circumscribe a $\odot$ about $\triangle ABC$.	1. Possible construction.
2. Extend AD to meet the $\odot$ at E. Draw CE.	2. Possible construction.
3. $bc = AE \cdot t_A$.	3. Give proof.
4. $\therefore bc = t_A(t_A + s)$	4. $AE = t_A + s$.
5. $\therefore bc = t_A^2 + s \cdot t_A$.	5. Algebra.
6. But $s \cdot t_A = rp$.	6. § 283.
7. $\therefore bc = t_A^2 + rp$.	7. Ax. 2 § 49.

320. Cor. $t_A^2 = bc - rp$, where b and c are sides of a triangle, t_A the bisector of the angle between them, and r and p the segments of the third side made by t_A.

Ex. 230. If $c = 4$, $b = 5$, and $a = 6$, find t_A.

Suggestions. 1. It is necessary to find r and p first. This may be done by using § 271.

$$\frac{r}{6 - r} = \frac{4}{5}. \quad \text{Whence } r = ? \text{ and } p = 6 - r = ?$$

2. Then substitute in the conclusion of § 320.

Note. Additional Exercises 119 to 122, p. 298, can be studied now.

BOOK IV

AREAS OF POLYGONS

321. A polygon, being a closed line (§ 15), **incloses a limited portion of the plane, called the interior of the polygon.**

322. The usual **unit of surface measure** is inclosed by a square whose side is a linear unit; as, a square inch, or a square foot.

The **area of the interior of a polygon** is the number of units of surface measure contained in it.

For convenience, the area of the interior of a rectangle, or triangle, or parallelogram, etc., is referred to as the area of the rectangle, the triangle, or parallelogram.

Ex. 1. In the following figures, assume that the unit of surface is a small square. (*a*) What is the exact area of Figs. 1 and 2?

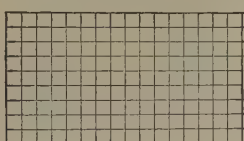


Fig. 1

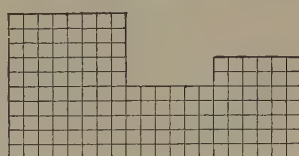


Fig. 2

(*b*) What is the approximate area of Figs. 3, 4, and 5?

(Include the square in the area if half or more than half of it lies within the figure; do not include it otherwise.)

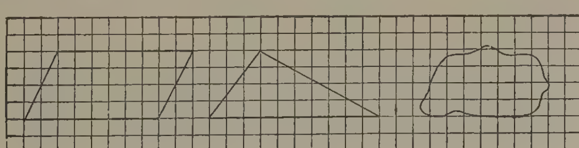


Fig. 3 Fig. 4 Fig. 5

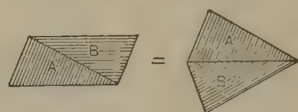
323. Two limited portions of a plane are **equal** if their areas are equal when they are measured by the same unit.

Some books call such figures *equivalent.*

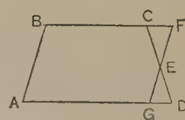
Since the test of the equality of two figures is the equality of two numbers, the usual axioms apply when equal figures are added or subtracted, or when they are multiplied or divided by the same number.

Thus, if equal figures are added to equal figures, the sums are equal; also halves of equal figures are equal.

324. *Two congruent figures are necessarily equal, but two equal figures are not necessarily congruent. Also, two figures which consist of parts which are respectively congruent are equal.*

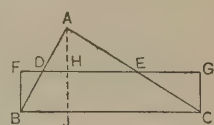


Thus, the parallelogram and the kite-shaped figure made from it by placing the two triangles together as in the figure adjoining are equal.



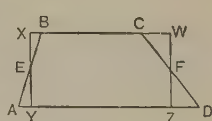
Ex. 2. If E is the mid-point of one of the non-parallel sides of trapezoid $ABCD$, and a parallel to AB drawn through E meets BC extended at F and AD at G, prove that parallelogram $ABFG$ is equal to trapezoid $ABCD$.

Suggestion. Prove $\triangle CEF \cong \triangle GED$, and apply § 324.



Ex. 3. In the adjoining figure, D and E are the mid-points of AB and AC; $AJ \perp BC$; $BF \perp DE$ extended at F; $CG \perp DE$ extended at G. Prove that $\triangle ABC$ equals $\square BFGC$.

Suggestion. Prove $\triangle BDF \cong \triangle DAH$, and $\triangle CEG \cong \triangle AEH$.

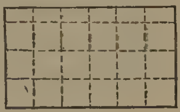


Ex. 4. In the adjoining figure, E and F are the mid-points of sides AB and CD of trapezoid $ABCD$; XY and ZW are drawn through E and F respectively $\perp AD$, meeting BC extended at X and W respectively. Prove that $XYZW = ABCD$.

Ex. 5. Let K be the mid-point of side BC and H the mid-point of side AD of $\square ABCD$; let FE, drawn through the mid-point G of KH, intersect BC and AD at F and E respectively. Prove that FE divides $ABCD$ into two equal quadrilaterals.

Note. Additional Exercises 1–2, p. 299, can be studied now.

325. Area of a rectangle. If the base of a rectangle measures 6 and its altitude 5 linear units, the area is evidently 6×5 or 30 surface units.



If the base measures 6 units and the altitude measures $3\frac{1}{2}$ units, the area is evidently 6×3.5 or 21 surface units.

These two examples suggest the theorem:

The number of surface units within a rectangle is the product of the number of linear units in its base and the number in its altitude. More briefly, this theorem is expressed as follows.

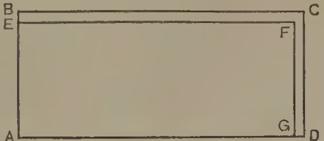
326. *The area of a rectangle is the product of its base and its altitude.*

This theorem will be taken as the *definition* of the area of a rectangle, since:

(*a*) it gives the area when the base and altitude are each expressible exactly in terms of a common linear unit; and

(*b*) it gives what is obviously the approximate area when the base or the altitude, or both, are expressible only approximately in terms of a common linear unit.

Thus, if AB is a "little" more than .6 in. and $AE = .6$ in.; if AD is a "little" more than 1.5 in. and $AG = 1.5$ in.; then $.6 \times 1.5$ or .90 sq. in., which is the area of $AEFG$, is obviously approximately the area of $ABCD$.



The theorem can be proved as the culmination of three theorems which are often given in texts.

If a = the number of linear units in the altitude, and b = the number of the same units in the base, and A = the number of the corresponding surface units in the area, then $A = ab$ **is the formula for the area of a rectangle.**

Ex. 6. What is the area, correct to tenths, of the rectangle whose altitude is 2.7 in. and whose base is 3.5 in.?

327. Comparison of rectangles.

(*a*) *Two rectangles having equal bases and equal altitudes are equal.*

If a_1 and a_2 are the altitudes, if b_1 and b_2 are the bases, and A_1 and A_2 are the areas, then $A_1 = a_1 \times b_1$ and $A_2 = a_2 \times b_2$.

If $a_1 = a_2$ and $b_1 = b_2$, then $a_1b_1 = a_2b_2$, or $A_1 = A_2$. (Ax. 5.)

(*b*) *Two rectangles are to each other as the products of their altitudes and their bases.*

Since $A_1 = a_1b_1$ and $A_2 = a_2b_2$, then $\frac{A_1}{A_2} = \frac{a_1b_1}{a_2b_2}$. (Ax. 6.)

Example. If the altitudes of two rectangles R and S are 5 in. and 4 in. respectively and their bases are 6 in. and 10 in. respectively, then

$$\frac{R}{S} = \frac{5 \times 6}{4 \times 10}, \text{ or } \frac{R}{S} = \frac{3}{4}.$$

(*c*) *Two rectangles having equal bases are to each other as their altitudes.*

For $\frac{A_1}{A_2} = \frac{a_1b_1}{a_2b_2}$. If $b_1 = b_2$, then $\frac{a_1b_1}{a_2b_2} = \frac{a_1}{a_2}$, or $\frac{A_1}{A_2} = \frac{a_1}{a_2}$.

Example. If two rectangles R and S have altitudes of 8 and 12 respectively, and the base 10, then $R : S = 8 : 12$, or $R : S = 2 : 3$.

(*d*) *Two rectangles having equal altitudes are to each other as their bases.*

For $\frac{A_1}{A_2} = \frac{a_1b_1}{a_2b_2}$. If $a_1 = a_2$, then $\frac{a_1b_1}{a_2b_2} = \frac{b_1}{b_2}$, or $\frac{A_1}{A_2} = \frac{b_1}{b_2}$.

Example. If two rectangles R and S have altitude 15 in. and bases 20 and 25 respectively, then $R : S = 20 : 25$, or $R : S = 4 : 5$.

Ex. 7. If the rectangles R, S, T, X, and Y have the dimensions indicated:

Find the ratio of R to T; of R to Y.

Ex. 8. Find the ratio of S to X; of S to Y.

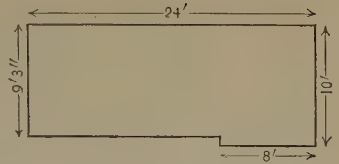
Ex. 9. Find the ratio of T to Y; of T to X.

Ex. 10. Find the ratio of X to R; of X to Y.

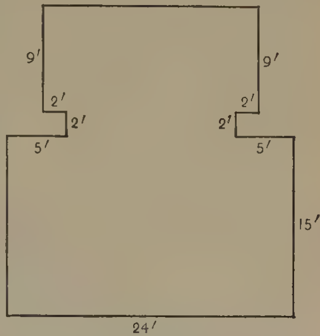
Rect.	Altitude	Base
R	10 in.	8 in.
S	5 in.	14 in.
T	10 in.	6 in.
X	12 in.	14 in.
Y	10 in.	14 in.

Ex. 11. A lot on a business street is 40 ft. wide and 150 ft. long. It sold for $20,000. How much was this per square foot?

Ex. 12. What is the cost of a concrete walk, 5 ft. wide, on the two sides of a corner lot which is 40 ft. wide and 150 ft. long at 30¢ per square foot?



Ex. 13. What is the cost of a concrete floor for a porch having the shape and dimensions shown in the adjoining figure, at 33¢ per square foot?



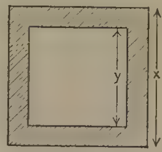
Ex. 14. What is the cost of a tile floor for a bath room 7 ft. 6 in. wide and 9 ft. long at 90¢ per square foot?

Ex. 15. What is the cost of a tile floor for a room having the shape and dimensions shown in the adjoining figure at 90¢ per square foot?

Ex. 16. What is the cost of decorating the ceiling of this same room at $1.00 per square yard?

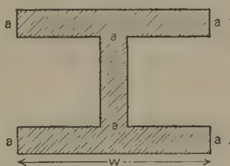
Ex. 17. (*a*) An acre contains 160 square rods. A rod is 16.5 ft. How many square feet are there in an acre?

(*b*) How long must the side of a square field be in order to contain one acre?



Ex. 18. (*a*) Make a formula for the area of the shaded surface lying between two squares having the dimensions indicated.

(*b*) Find the area when $x = 12$ in. and $y = 9$ in.



Ex. 19. (*a*) Make a formula for the area of the shaded surface shown in the adjoining figure.

(*b*) Find the area when $a = 2$ in., $b = 12$ in., and $w = 10$ in.

Ex. 20. The area of a rectangle is 147 sq. in. Its base is three times its altitude. What are its dimensions?

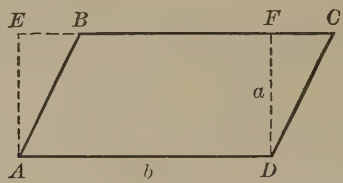
Ex. 21. What is the length of the diagonal of a rectangle whose area is 2640 sq. ft. if its altitude is 48 ft.?

Ex. 22. How many bricks having an exposed surface of 2 in. by 8 in. will it take to cover a floor 10 ft. wide and 22 ft. long?

Note. Additional Exercises 3–8, p. 299, can be studied now.

* PROPOSITION I. THEOREM

328. *The area of a parallelogram equals the product of its base and altitude.*



Hypothesis. $ABCD$ is a parallelogram.
Its altitude $DF = a$; its base $AD = b$.

Conclusion. Area of $ABCD = ab$.

Plan. Change $\square ABCD$ into an equal rectangle.

Proof:

STATEMENTS	REASONS
1. Draw $AE \parallel DF$, meeting CB extended at E.	1. Why possible.
2. $AEFD$ is a rectangle.	2. Give the proof.
3. Area $\square AEFD = ab$.	3. Why?
4. $\triangle AEB \cong \triangle DFC$.	4. Give the proof.
5. $\therefore \square ABCD = \square AEFD$.	5. Give the proof.
6. $\therefore$ area $\square ABCD = ab$.	6. Why?

329. Corollaries. Let $\square P_1$ have base b_1 and altitude a_1; and $\square P_2$ have base b_2 and altitude a_2.

I. *Parallelograms having equal bases and equal altitudes are equal.*

II. *Two parallelograms are to each other as the products of their bases and their altitudes.*

For, since $\square P_1 = a_1b_1$ and $\square P_2 = a_2b_2$, then $\dfrac{\square P_1}{\square P_2} = \dfrac{a_1b_1}{a_2b_2}$.

III. *Parallelograms having equal altitudes are to each other as their bases.*

For, in (II), if $a_1 = a_2$, then $\square P_1 : \square P_2 = b_1 : b_2$.

IV. *Parallelograms having equal bases are to each other as their altitudes.*

Ex. 23. What is the area of ▱R, of ▱S, and of ▱T?

(*a*) ▱R has altitude 4 in. and base 9 in.

(*b*) ▱S has altitude 15 ft. and base 20 ft.

(*c*) ▱T has altitude $3x$ in. and base $11y$ in.

Ex. 24. What is the altitude of a parallelogram whose area is 56 sq. in., if its base is 14 in.?

Ex. 25. Construct a parallelogram equal to twice a given parallelogram.

Ex. 26. Construct a rectangle equal to two thirds a given parallelogram.

Ex. 27. Divide a parallelogram into four equal parallelograms by lines parallel to one side.

Ex. 28. What is the ratio of ▱P to ▱R if the base of each is 10 in. and the altitudes are 5 in. and 8 in. respectively?

Ex. 29. (*a*) Construct a ▱$ABCD$ having $AB = 3$ in. and $BC = 4$ in., and having $\angle B = 30°$; determine the area of the parallelogram.

(*b*) Repeat the exercise when $\angle B = 45°$.

(*c*) Repeat the exercise when $\angle B = 60°$.

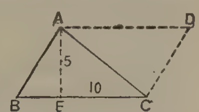
(*d*) How does the area change when the sides remain constant (unchanged) and included $\angle B$ increases?

Ex. 30. (*a*) On line AB, place C and D so that $CD = 2$ in. At C, construct CF, perpendicular to AB, making $CF = 1$ in. Through F, construct XY, perpendicular to CF. At D, construct DE, perpendicular to AB, meeting XY at E. What is the area of $CDEF$?

(*b*) With C as center and a radius of 2.5 in., draw an arc cutting XY at L; from D, draw a parallel to CL, meeting XY at M. Compare figure $CDML$ with $CDEF$. What is the area of $CDML$?

(*c*) With C as center and a radius of 2 in. draw an arc cutting XY at H; from D, draw DK, parallel to CH, meeting XY at K. What kind of figure is $CDKH$? What is its area?

(*d*) What is true about all parallelograms having the same base on one of two parallel lines, and their opposite sides on the other parallel?

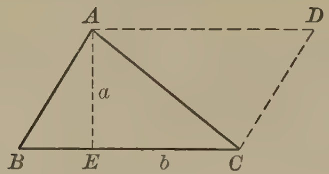


Ex. 31. The base of $\triangle ABC$ is 10 and the altitude is 5. What is the area of $\triangle ABC$?

Suggestion. Draw $AD \parallel BC$ and $CD \parallel AB$ to form ▱$ABCD$. Compare $\triangle ABC$ with ▱$ABCD$. Then determine the area of ▱$ABCD$ and finally of $\triangle ABC$.

PROPOSITION II. THEOREM

330. *The area of a triangle equals one half the product of its base and altitude.*



Hypothesis. $\triangle ABC$ has altitude $AE = a$ and base $BC = b$.
Conclusion. Area of $\triangle ABC = \frac{1}{2} ab$.
Plan. Compare $\triangle ABC$ with a related parallelogram.
(Construction and proof suggested by the figure and Ex. 31, p. 209.)

331. Corollaries. By proofs similar to those in § 329:

I. *Triangles having equal bases and equal altitudes are equal.*

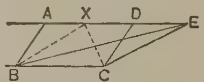
II. *Two triangles are to each other as the products of their bases and their altitudes.*

III. *Triangles having equal altitudes are to each other as their bases.*

IV. *Triangles having equal bases are to each other as their altitudes.*

V. *A triangle is one half a parallelogram having the same base and altitude.*

Ex. 32. (*a*) Compare $\square ABCD$ with $\triangle BCE$.
(*b*) Compare $\triangle BCX$ with $\triangle BCE$.
(*c*) If X is the mid-point of AD, compare $\triangle ABX$ with $\triangle XCD$; also compare $\triangle XCD$ with $\triangle BCE$.



Ex. 33. Find the area of an isosceles right triangle whose leg is 9 in.

Ex. 34. What is the length of the side of a square whose area equals that of a triangle whose base is 24 and whose altitude is 12?

Ex. 35. What is the area of the rhombus whose diagonals are 10 and 16 respectively?

Note. Additional Exercises 9 to 20, p. 299, can be studied now.

332. The area of a triangle expressed in terms of its sides.

Solution. 1. If a, b, and c are the sides of $\triangle ABC$, and $s = \frac{1}{2}(a + b + c)$, it can be proved that the altitude drawn to side a is given by the formula

$$h_a = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}. \qquad (\S\ 314)$$

2. $$\text{Area of } \triangle ABC = \tfrac{1}{2}\, a \times h_a.$$

3. $$\therefore \text{ area of } \triangle ABC = \frac{1}{2}\, a \times \frac{a}{2}\sqrt{s(s-a)(s-b)(s-c)}.$$

4. $$\therefore \text{ area of } \triangle ABC = \sqrt{s(s-a)(s-b)(s-c)}.$$

Example. Find the area of the triangle whose sides are 13, 14, and 15.

Solution. 1. Let $a = 13$, $b = 14$, and $c = 15$.

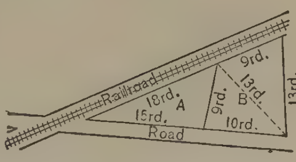
2. $\therefore s = \frac{1}{2}(13 + 14 + 15)$, or $s = 21$.

3. $\therefore$ area of the $\triangle = \sqrt{21 \times 8 \times 7 \times 6}$,
or $\sqrt{3 \times 7 \times 2 \times 4 \times 7 \times 2 \times 3}$.

4. $\therefore$ area of the $\triangle = 3 \times 7 \times 2 \times 2$, or 84.

Ex. 36. Find the areas of the triangles whose sides are:
(*a*) 12, 16, and 20; (*b*) 15, 18, and 21; (*c*) 20, 30, and 40.

Ex. 37. The sides of the lots A and B in the adjoining figure have the lengths indicated. Find the area of each lot.



Ex. 38. By the formula of § 332, find the area of the equilateral triangle whose side is 14 in. long.

Ex. 39. (*a*) By the same formula, prove that the area of the equilateral triangle whose side is a is $\frac{a^2}{4}\sqrt{3}$. (Memorize this formula.)

(*b*) By this new formula, find the area of the equilateral triangle whose side is: (*a*) 12 in.; (*b*) 15 in.; (*c*) 20 in.

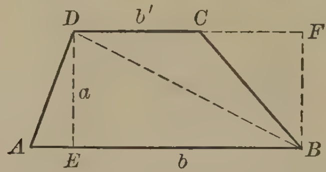
Ex. 40. If BD is the median to side AC of $\triangle ABC$, prove that $\triangle ABD = \triangle BDC$. (Draw the altitude to AC.)

Ex. 41. Prove that the diagonals of a parallelogram divide the parallelogram into four equal triangles.

Note. Additional Exercises 21 to 30, p. 300, can be studied now.

* PROPOSITION III. THEOREM

333. *The area of a trapezoid equals one half its altitude multiplied by the sum of its bases.*



Hypothesis. Trapezoid $ABCD$ has its altitude $DE = a$, its base $AB = b$, and its base $CD = b'$.

Conclusion. Area $ABCD = \frac{1}{2}a(b + b')$.

Proof: Statements	Reasons
1. Draw BD and alt. BF of $\triangle BCD$.	1. Why possible?
2. $ABCD = \triangle ABD + \triangle BCD$.	2. Why?
3. Area $\triangle ABD = \frac{1}{2}ab$.	3. Why?
4. Area $\triangle BCD = \frac{1}{2}ab'$.	4. Give the full proof.
5. $\therefore$ area $ABCD = \frac{1}{2}ab + \frac{1}{2}ab'$.	5. Why?
6. $\therefore$ area $ABCD = \frac{1}{2}a(b + b')$.	6. Factoring.

334. Cor. *The area of a trapezoid equals the product of its altitude and its median.*

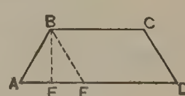
Suggestion. Recall § 161.

Ex. 42. Determine the area of the trapezoid whose altitude is 10 in., and whose bases are 9 in. and 20 in.

Ex. 43. Find the lower base of a trapezoid whose area is 675 sq. ft., if the upper base is 35 ft., and altitude 15 ft.

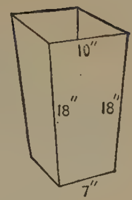
Suggestion. Let x = the No. of ft. in the lower base.

Ex. 44. The non-parallel sides, AB and CD, of a trapezoid are each 25 in., and the sides AD and BC are 33 in. and 19 in., respectively. Find the area of the trapezoid.



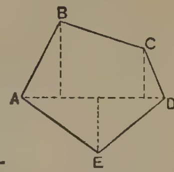
Suggestions. Draw through B a $\parallel$ to CD, and a $\perp$ to AD.

Ex. 45. Prove that the straight line joining the mid-points of the bases of a trapezoid divides the trapezoid into two equal trapezoids.



Ex. 46. How many square feet of wood will be required for the sides of 100 waste-paper boxes like the one pictured in the adjoining figure, — allowing 15% extra for wood wasted in cutting?

Note. Assume that each side is an isosceles trapezoid having the dimensions indicated in the figure.



Ex. 47. The longest diagonal AD of pentagon $ABCDE$ is 44 in., and the perpendiculars to it from B, C, and E are 24, 16, and 15 in. respectively. If $AB = 25$ in. and $CD = 20$ in., what is the area of the pentagon?

Ex. 48. Construct a triangle equal to a given trapezoid and having the same altitude as the trapezoid.

Ex. 49. Find the lower base of a trapezoid whose area is 9408 sq. ft., whose upper base is 79 ft., and whose altitude is 96 ft.

Ex. 50. Draw through a given point in one side of a parallelogram a straight line, dividing the parallelogram into two equal parts.

Ex. 51. Construct a parallelogram equal to a given trapezoid, having the same altitude as the trapezoid.

Ex. 52. If AD is the median to side BC of $\triangle ABC$ and E is the mid-point of AD, then $\triangle BEC = \frac{1}{2}\triangle ABC$.

Ex. 53. If E and F are the mid-points of sides AB and AC respectively of $\triangle ABC$, and D is any point in side BC, prove quadrilateral $AEDF = \frac{1}{2}\triangle ABC$.

Ex. 54. If E is any point in side BC of $\square ABCD$, then $\triangle ABE + \triangle ECD = \frac{1}{2}\square ABCD$.

Ex. 55. Draw a straight line perpendicular to the bases of a trapezoid which will divide the trapezoid into two equal parts.

Ex. 56. What is the area of the trapezoid $ABCD$ whose bases DC and AB are 14 in. and 20 in., if $AD = 12$ in. and $\angle A = 30°$?

Ex. 57. Repeat Ex. 56 if $\angle A = 60°$.

Ex. 58. Repeat Ex. 56 if $\angle A = 45°$.

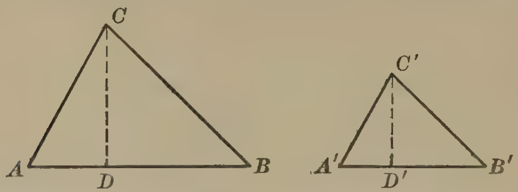
Ex. 59. What is the area of the isosceles trapezoid whose bases are 12 in. and 20 in. respectively, and whose non-parallel sides make angles of 45° with its lower base?

Ex. 60. Repeat Ex. 59 when the non-parallel sides make with the lower base an angle of 60°.

Note. Additional Exercises 31 to 48, p. 301, can be studied now.

* PROPOSITION IV. THEOREM

335. *The areas of two similar triangles are to each other as the squares of any two corresponding sides.*



Hypothesis. AB and $A'B'$ are corresponding sides of similar $\triangle ABC$ and $A'B'C'$ respectively.

Conclusion. $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2} = \dfrac{\overline{BC}^2}{\overline{B'C'}^2} = \dfrac{\overline{AC}^2}{\overline{A'C'}^2}$.

Proof:

STATEMENTS	REASONS
1. Draw altitudes CD and $C'D'$.	1. Why possible?
2. $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{\frac{1}{2} AB \cdot CD}{\frac{1}{2} A'B' \cdot C'D'}$, or $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{AB \cdot CD}{A'B' \cdot C'D'}$.	2. § 330; Ax. 6, § 49.
3. $\therefore \dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{AB}{A'B'} \cdot \dfrac{CD}{C'D'}$.	3. Ax. 2, § 49.
4. But $\dfrac{CD}{C'D'} = \dfrac{AB}{A'B'}$.	4. § 282.
5. $\therefore \dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$.	5. Ax. 2, § 49.
6. But $\dfrac{BC}{B'C'} = \dfrac{AC}{A'C'} = \dfrac{AB}{A'B'}$.	6. Why?
7. $\therefore \dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{\overline{BC}^2}{\overline{B'C'}^2} = \dfrac{\overline{AC}^2}{\overline{A'C'}^2}$.	7. Ax. 2, § 49.

Ex. 61. $\triangle ABC \sim \triangle A'B'C'$ and $AB = 2\,A'B'$.

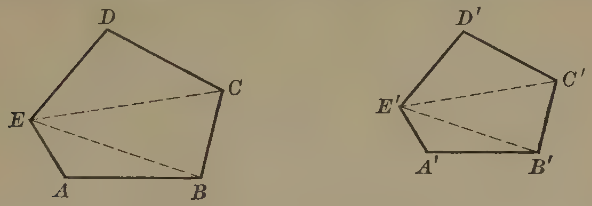
(*a*) Compare the area of $\triangle ABC$ with the area of $\triangle A'B'C'$.

(*b*) Draw a figure to illustrate the correctness of your result.

Note. Additional Exercises 49 to 67, p. 302, can be studied now.

* PROPOSITION V. THEOREM

336. *The areas of two similar polygons are to each other as the squares of any two corresponding sides.*



Hypothesis. AB and $A'B'$ are corresponding sides of similar polygons AC and $A'C'$.

Conclusion. $\dfrac{\text{Area of polygon } AC}{\text{Area of polygon } A'C'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$, etc.

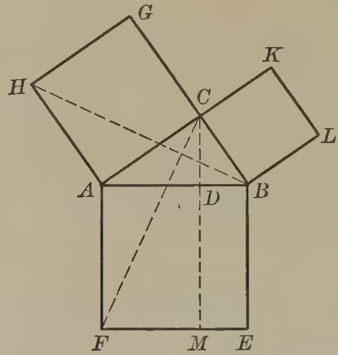
Proof:

STATEMENTS	REASONS
1. Draw EB, EC, $E'B'$, and $E'C'$.	1. Why possible?
2. $\therefore \triangle ABE \sim \triangle A'B'E'$; $\triangle EBC \sim \triangle E'B'C'$; etc.	2. § 295.
3. $\dfrac{\triangle ABE}{\triangle A'B'E'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$.	3. § 335.
4. Also $\dfrac{\triangle BCE}{\triangle B'C'E'} = \dfrac{\overline{BC}^2}{\overline{B'C'}^2} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$.	4. Why?
5. Similarly $\dfrac{\triangle CDE}{\triangle C'D'E'} = \dfrac{\overline{CD}^2}{\overline{C'D'}^2} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$.	5. Step 4.
6. $\therefore \dfrac{\triangle ABE}{\triangle A'B'E'} = \dfrac{\triangle BCE}{\triangle B'C'E'} = \dfrac{\triangle CDE}{\triangle C'D'E'}$.	6. Ax. 1, § 49.
7. $\therefore \dfrac{\triangle ABE + \triangle BCE + \triangle CDE}{\triangle A'B'E' + \triangle B'C'E' + \triangle C'D'E'} = \dfrac{\triangle ABE}{\triangle A'B'E'}$.	7. § 296.
8. $\therefore \dfrac{\text{polygon } ABCDE}{\text{polygon } A'B'C'D'E'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$, etc.	8. Ax. 1, § 49.

Note. Additional Exercises 68 to 74, p. 303, can be studied now.

PROPOSITION VI. THEOREM

337. *The square upon the hypotenuse of a right triangle is equal to the sum of the squares upon the two legs of the triangle.*



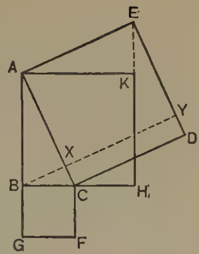
Hypothesis. $\angle C$ of $\triangle ABC$ is a right angle.
$ABEF$, $ACGH$, and $BCKL$ are squares.

Conclusion. Area $ABEF$ = area $ACGH$ + area $BCKL$.

Proof:

STATEMENTS	REASONS
1. Draw $CD \perp AB$, meeting FE at M, when extended. Draw BH and CF.	1. Why possible?
2. $\triangle ACF \cong \triangle ABH$.	2. Give the proof.
3. BCG is a straight line.	3. § 41.
4. $\triangle ABH$ and $\square ACGH$ have the same base, AH, and equal altitudes.	4. Prove the altitudes are equal.
5. $\therefore$ area $ACGH = 2$ area $\triangle ABH$.	5. Why?
6. Also area $ADMF = 2$ area $\triangle ACF$.	6. Give the proof.
7. $\therefore$ area $ACGH$ = area $ADMF$.	7. Why?
8. Similarly it can be proved that area $BCKL$ = area $BDME$.	8. As in Steps 2–7.
9. $\therefore$ area $ACGH$ + area $BCKL$ = area $ABEF$.	9. Why?

Note. This purely geometrical proof of the Pythagorean Theorem is attributed to Euclid. Pythagoras' own proof is unknown.



Ex. 62. Prove the Pythagorean Theorem, using the adjoining figure. (Note. Square AH is "turned in" over $\triangle ABC$.)

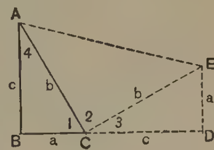
Prove $\square AD = \square BF + \square AH$.

Suggestions. 1. Draw EK and prove HKE is a st. line, by proving $\angle AKE$ is a rt. $\angle$.

2. Prove $\square AH = \square AXYE$, by comparing each with $\triangle ABE$.

3. Prove $\square BF = \square CXYD$.

Note. The Pythagorean Theorem can be proved from figures obtained by "turning in" any of the squares, one at a time, two at a time, or all three of them.



Ex. 63. Garfield's Proof of the Pythagorean Theorem.

Hyp. In $\triangle ABC$, $\angle B = 90°$.

Con. $b^2 = a^2 + c^2$.

Suggestions. 1. Extend BC to D, making $CD = AB$. Draw $DE \perp BD$ at D, making $DE = BC$. Draw CE and AE.

2. Prove $ABDE$ is a trapezoid.

3. Express the area of $ABDE$ in terms of a and c.

4. Prove $\angle 2 = 90°$, and that $CE = b$.

5. Express the area of $\triangle ABC$, CDE, ACE in terms of a, b, and c.

6. Form an equation based on the fact that the trapezoid consists of the three triangles.

7. Complete the proof algebraically.

Ex. 64. Prove C, H, and L lie in a st. line. (Fig. § 337.)

(Draw CH and CL, and prove $\angle HCL = 1$ st. $\angle$.)

Ex. 65. Prove $AG \parallel BK$.

Ex. 66. Prove that the sum of the $\perp$s from H and L to AB extended equals AB.

Suggestion. Compare AD and DB with the $\perp$s.

Ex. 67. Prove $\triangle GCK = \triangle ABC$.

Ex. 68. Prove $\triangle LBE = \triangle ABC$.

Suggestion. Draw altitude LJ to EB extended.

Note 1. A number of alternative proofs of the Pythagorean Theorem, and other interesting theorems, are given in Heath's Mathematical Monographs, Numbers 1–4. Published by D. C. Heath & Co., Boston, New York, Chicago, Atlanta, San Francisco.

Note 2. Additional Exercises 75 to 78, p. 304, can be studied now.

Miscellaneous Review

Ex. 69. Through the vertex A of $\triangle ABC$, draw a line MN parallel to BC. On MN take any point X and prove that $\triangle XBC = \triangle ABC$.

Ex. 70. Construct a triangle twice as large as a given triangle:
(a) having the same base as the given triangle;
(b) having the same altitude as the given triangle.

Ex. 71. Construct a rectangle equal to a given triangle.

Ex. 72. Construct a triangle equal to a given rectangle.

Ex. 73. Construct a right triangle equal to a given triangle and having the same base as the triangle.

Ex. 74. Construct an isosceles triangle equal to a given triangle and having the same base as the given triangle.

Ex. 75. Construct a square equal to the sum of two given squares.

Suggestion. Base the construction on the Pythagorean Theorem.

Ex. 76. Construct a square equal to the difference of two given squares.

Ex. 77. Construct a square equal to the sum of three given squares.

Ex. 78. The area of an isosceles right triangle is equal to one fourth the area of the square described upon the hypotenuse.

Suggestion. Compare the right triangle with the square on one leg.

Ex. 79. What is the ratio of $\triangle ABC$ to $\triangle A'B'C'$, if they are similar, and: (a) if $AB = 3\,A'B'$? (b) if $AB = A'B'$? (c) if $AB = \frac{2}{3}\,A'B'$?

Ex. 80. If the area of a polygon, one of whose sides is 5 in., is 375 sq. in., what is the area of a similar polygon whose corresponding side is 8 in.?

Ex. 81. The longest sides of two similar polygons are 18 in. and 3 in. respectively. How many similar polygons, each equal to the second, will form a polygon equal to the first?

Ex. 82. Draw through a given point in one base of a trapezoid a straight line which will divide the trapezoid into two equal parts.

Ex. 83. If the diagonals of a quadrilateral are perpendicular, the sum of the squares on one pair of opposite sides of the quadrilateral equals the sum of the squares on the other pair.

Ex. 84. Prove that two similar triangles are to each other as the squares of any two corresponding altitudes.

Ex. 85. The sides AB, BC, CD, and DA of quadrilateral $ABCD$ are 10, 17, 13, and 20 respectively, and the diagonal AC is 21. Find the area of the quadrilateral.

Ex. 86. If diagonals AC and BD of trapezoid $ABCD$ intersect at E, then $\triangle AEB = \triangle DEC$. ($BC$ and AD are the bases of $ABCD$.)

Suggestion. Compare $\triangle ABD$ and $\triangle ACD$.

Ex. 87. If X is any point in diagonal AC of $\square ABCD$, then

$$\triangle ABX = \triangle AXD.$$

Suggestion. Draw the altitudes from B and D to base AX.

Ex. 88. If E and F are the mid-points of sides AB and AC of $\triangle ABC$, then $\triangle AEF = \frac{1}{4}\triangle ABC$.

Ex. 89. If E is any point within $\square ABCD$, then $\triangle ABE + \triangle CDE$ equals $\frac{1}{2}$ the parallelogram.

Suggestion. Through E draw a line parallel to AB.

Ex. 90. If $\angle A$ of $\triangle ABC$ is 30°, prove that the area of $\triangle ABC = \frac{1}{4} AB \times AC$.

Ex. 91. Prove that the area of a rhombus is one half the product of its diagonals.

Ex. 92. If E is the mid-point of CD, one of the non-parallel sides of trapezoid $ABCD$, prove that $ABE = \frac{1}{2} ABCD$.

Suggestion. Through E, draw a line parallel to AB.

Ex. 93. From one vertex of a parallelogram, draw lines dividing the parallelogram into three equal parts.

Ex. 94. Define area of a plane figure.

Ex. 95. Distinguish between congruent, similar, and equal figures.

Ex. 96. State the formula for the area of any triangle in terms of its sides a, b, and c, and the number s. What is the number s?

Ex. 97. (a) What is the formula for the area of an equilateral triangle in terms of its side a?

(b) By the formula for the area of an equilateral triangle in terms of its side a, find the area of the equilateral $\triangle$ with side 8 in.

Ex. 98. State the corollaries by which the areas of two rectangles, or two parallelograms, or two triangles are compared:

(a) If they have equal altitudes.

(b) If they have equal bases.

(c) When no known relation exists between the altitudes or the bases.

Ex. 99. State the theorem connecting the areas of a triangle and a parallelogram having equal bases and equal altitudes.

Ex. 100. State the theorem connecting the areas of two similar polygons.

OPTIONAL TOPICS

Three optional topics follow, — optional in the sense that they do not appear explicitly in the lists of the College Entrance Examination Board or the National Committee. As stated previously, while failure to include them does not imply either disbelief in their educational value or in their mathematical interest, it does justify omission of them from the minimum essentials course. On the other hand, it will be distinctly unfortunate to fail to allow at least the good and the excellent pupils the pleasure and the profit of studying some or all of this material.

All of this material appears in some form in geometries; none of it is necessary as preparation for the study of the main parts of subsequent geometry.

Topic A. **Constructions Based on Algebraic Analysis.**

One of the most fruitful opportunities to correlate algebra and geometry, and, as taught in this text, quite easy. Pages 221–223.

Topic B. **Constructions Based on Geometric Analysis.**

Further practice in the kind of analysis taught first on pages 143 to 150. These pages are not a necessary preparation for the study of Topic B, however. Pages 224–225.

Topic C. **Miscellaneous Construction Problems.**

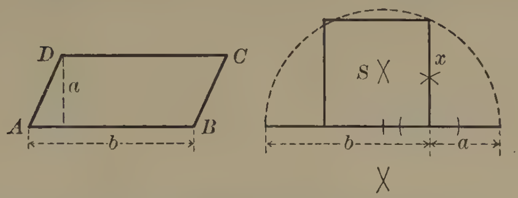
A number of interesting problems which have appeared in all geometries, but which can safely be omitted from a minimum course. They all have special interest historically and therefore merit the attention of at least the excellent pupils of a class. Pages 226–227.

Optional Topic A

CONSTRUCTIONS BASED ON ALGEBRAIC ANALYSIS

PROPOSITION VII. PROBLEM

338. *Construct a square equal to a given parallelogram.*



Given $\square ABCD$ having base b and altitude a.

Required to construct a square equal to $\square ABCD$.

Analysis. 1. Let x = the side of the required square.	
2. Then x^2 = the area of the required square,	Why?
and ab = the area of the given parallelogram.	Why?
3. $\therefore x^2 = ab$.	Why?
4. $\therefore a : x = x : b$.	§ 252
5. $\therefore$ x is the mean proportional between a and b, and can be constructed by § 289.	§ 251
Construction. 1. Construct x, the mean proportional between a and b.	§ 289
2. On x as side, construct the square S.	
Statement. Square $S = \square ABCD$.	
Proof. 1. $a : x = x : b$, or $x^2 = ab$.	Why?
2. But area of $S = x^2$,	
and area of $\square ABCD = ab$.	
3. $\therefore$ area of S = area of $\square ABCD$.	
Discussion. The construction is always possible.	

Note. The analysis gives the pupil an idea of how such a construction is discovered. In most cases the proof of the correctness of the resulting construction is rather trivial after such an algebraic analysis, — and in such cases the teacher may properly omit the proof.

339. Cor. *Construct a square equal to a given triangle.*

Analysis. 1. Let x = the side of the required square, b = the base of the given triangle, and h = the altitude of the triangle.

2. $\therefore x^2 = \frac{1}{2} hb.$

3. $\therefore \frac{1}{2} b : x = x : h$, or x is the mean proportional between $\frac{1}{2} b$ and h.

Construction to be given by the pupil.

Ex. 101. Construct a square equal to twice a given triangle.

Ex. 102. Construct a square equal to twice a given square.

Ex. 103. Construct a square which will be twice a given parallelogram.

Ex. 104. Construct a square which will be three times a given triangle.

Ex. 105. (*a*) Construct a square which will be two thirds a given rectangle.

(*b*) Construct a square which will equal twice a given trapezoid.

Ex. 106. Construct a parallelogram which will equal a given rectangle and have a given segment as base.

Analysis. 1. Let a = the altitude and b = the base of the given rectangle, and let c = the given base of the parallelogram.

Let x = the required altitude of the parallelogram.

2. Then $cx = ab.$ Why?

3. $\therefore c : a = b : x$, or x is the 4th proportional to c, a, and b. Why?

Construction left to the pupil.

Suggestion. Construct the fourth proportional x and then construct the ▱.

Ex. 107. (*a*) Construct a rectangle equal to a given rectangle, having a given segment as base. (Analyze as in Ex. 106.)

(*b*) Construct a rectangle equal to a given triangle, having a given segment as base.

Ex. 108. Construct a triangle equal to a given triangle, having a given segment as base.

Suggestion. Determine the altitude as in Ex. 106, then construct the triangle. How many such triangles can be constructed?

Ex. 109. Construct a rectangle having a given base and equal to $\frac{2}{3}$ a given square. (Analyze as in Ex. 106.)

Ex. 110. Construct a triangle having a given base and equal to a given parallelogram.

Ex. 111. Construct a parallelogram having a given altitude and equal to a given triangle.

Ex. 112. Construct a parallelogram having a given altitude and equal to a given square.

Ex. 113. Construct a parallelogram having a given altitude and equal to a given trapezoid.

Ex. 114. Construct a triangle having a given altitude and equal to a given trapezoid.

Ex. 115. Construct a right triangle having a given altitude and equal to a given triangle.

Ex. 116. (*a*) Construct a right triangle having a given base and equal to a given square.

(*b*) Construct an isosceles triangle having a given base and equal to a given square.

Ex. 117. Construct an isosceles triangle having a given altitude and equal to twice a given trapezoid.

Ex. 118. Construct an isosceles triangle having a given base and equal to one half a given parallelogram.

Ex. 119. Construct a rectangle having a given altitude and equal to a given parallelogram.

Ex. 120. Construct a rectangle having a given base and equal to a given trapezoid.

Ex. 121. Construct a right triangle having a given hypotenuse and equal to a given square.

Suggestion. After determining the altitude to the hypotenuse, construct the right triangle by using § 238.

Ex. 122. Construct a right triangle having a given hypotenuse and equal to twice a given triangle.

Ex. 123. (*a*) Construct a square equal to the sum of two given triangles.

Suggestion. First construct squares equal to the two triangles.

(*b*) Construct a square equal to the difference of two given triangles.

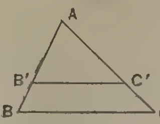
Ex. 124. Construct a line parallel to the base of a triangle which will divide the triangle into two equal parts.

Analysis. **1.** Assume $B'C'$ is the required line; let $AB' = x$.

2. $\therefore \dfrac{\triangle ABC}{\triangle AB'C'} = \dfrac{2}{1}$; and $\dfrac{\triangle ABC}{\triangle AB'C'} = \dfrac{AB^2}{x^2}$.

3. $\therefore \dfrac{AB^2}{x^2} = \dfrac{2}{1}$, or $x^2 = \frac{1}{2}\overline{AB}^2$.

4. $\therefore \frac{1}{2}\overline{AB} : x = x : \overline{AB}$.



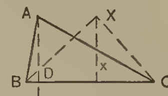
(Complete the analysis, and the construction.)

Optional Topic B

CONSTRUCTIONS BASED ON GEOMETRIC ANALYSIS

340. Transforming triangles and quadrilaterals means to construct a triangle or a quadrilateral, the area of which has some specified relation to the area of the given figure.

Ex. 125. Given $\triangle ABC$. Construct an equal triangle which has the same base BC and, at B, an angle of 60°.



Analysis. 1. Let $\triangle XBC$ represent the required $\triangle$.

2. Since $\triangle XBC = \triangle ABC$, and has the same base, the altitude from X equals altitude AD. (§ 331, IV.)

$\therefore$ the locus of X is a parallel to BC at the distance AD from it. (§ 240, 2.)

3. XB must be drawn so that $\angle XBC = 60°$.

Ex. 126. Construct a $\triangle XBC$ equal to a given $\triangle ABC$, having the same base BC and side XB equal to a given segment d.

Ex. 127. Construct a $\triangle XBC$ equal to a given $\triangle ABC$, having the same base BC, and having the median from X to BC equal to a given segment m.

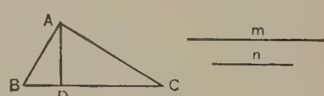
Ex. 128. Construct a $\square$ $XBCY$ equal to a given $\square ABCD$, having the same base BC and:

(*a*) having $\angle XBC$ = a given angle;

(*b*) having side XB = a given segment;

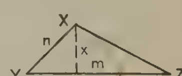
(*c*) having diagonal YB = a given segment.

Ex. 129. Construct a triangle equal to a given $\triangle ABC$ and having two of its sides equal to given segments m and n.



Given $\triangle ABC$ and segments m and n.

Required to construct a $\triangle$ equal to $\triangle ABC$, having m and n as two sides.



Analysis. 1. Let $\triangle XYZ$ represent the required $\triangle$.

2. Since $\triangle ABC = \triangle XYZ$, $AD \cdot BC = m \cdot x$ (§ 331, II).

or $m : AD = BC : x$. (§ 252)

3. $\therefore$ x is the fourth proportional to m, AD, and BC.

4. The locus of X is a parallel to m at the distance x from it.

5. X also is at the distance n from Y.

Ex. 130. Construct a triangle equal to a given square, having given its base and an angle adjacent to the base.

Ex. 131. Construct a triangle equal to a given triangle and having one side equal to a given segment m, and one angle adjacent to that side equal to a given angle, $\angle T$.

Ex. 132. Construct a triangle equal to a given square, having given its base and the median to the base.

Ex. 133. Construct a rhombus equal to a given parallelogram, having one of its diagonals coincident with a diagonal of the parallelogram.

Ex. 134. Given the base a, and $\angle B$ of a triangle, construct the triangle so that it will equal a given parallelogram.

Ex. 135. Given the base a and the $\angle B$ of a triangle, construct the triangle so that it will equal a given trapezoid.

Ex. 136. Given the sides a and b of a triangle, construct the triangle so that it will equal a given square.

Ex. 137. Given the side a and the median m_a to it, construct the triangle so that it will equal a given parallelogram.

Ex. 138. Given the side a of a right triangle, construct the triangle so that it will equal a given square.

Ex. 139. Given the side a of a right triangle, construct the triangle so that it will equal a given parallelogram.

Ex. 140. Construct a right triangle having a given hypotenuse so that it will equal a given square.

Ex. 141. Construct a right triangle having a given hypotenuse so that it will equal a given parallelogram.

Ex. 142. Construct a right triangle having a given hypotenuse so that it will equal a given trapezoid.

Ex. 143. Construct a parallelogram having a given base and a given diagonal so that it will equal a given square.

Ex. 144. Construct a parallelogram having a given base and a given diagonal so that it will equal a given trapezoid.

Ex. 145. Construct a parallelogram having a given base, and a given base angle, so that it will equal a given square.

Ex. 146. Construct a parallelogram having two given sides so that it will equal a given square.

Ex. 147. Construct a parallelogram having two given sides so that it will equal a given trapezoid.

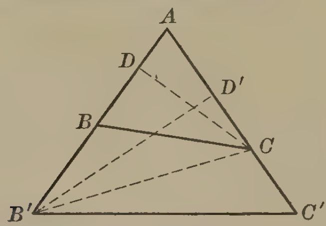
Ex. 148. Construct a trapezoid having given its two bases and one side, so that it will equal a given square.

Optional Topic C

MISCELLANEOUS THEOREMS AND PROBLEMS

PROPOSITION VIII. THEOREM

341. *Two triangles having an angle of one equal to an angle of the other are to each other as the products of the sides including these angles.*



Hypothesis. $\triangle ABC$ and $\triangle AB'C'$ have $\angle A$ common.

Conclusion. $\dfrac{\triangle ABC}{\triangle AB'C'} = \dfrac{AB \times AC}{AB' \times AC'}$.

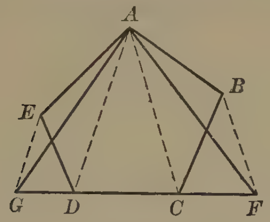
Proof: Statements	Reasons
1. Draw $B'C$, and alt. CD to AB'.	1. Why possible?
2. $\triangle ABC$ and $\triangle AB'C$ have alt. CD.	2. Definition of alt.
3. $\therefore \dfrac{\triangle ABC}{\triangle AB'C} = \dfrac{AB}{AB'}$.	3. § 331, III.
4. $\triangle AB'C$ and $\triangle AB'C'$ have as altitude, $B'D'$, the $\perp$ from B' to AC'.	4. Definition of alt.
5. $\therefore \dfrac{\triangle AB'C}{\triangle AB'C'} = \dfrac{AC}{AC'}$.	5. Why?
6. Multiplying the equations 3 and 5, $\dfrac{\triangle ABC}{\triangle AB'C} \cdot \dfrac{\triangle AB'C}{\triangle AB'C'} = \dfrac{AB \times AC}{AB' \times AC'}$.	6. Ax. 5, § 49.
7. $\therefore \dfrac{\triangle ABC}{\triangle AB'C'} = \dfrac{AB \times AC}{AB' \times AC'}$.	7. Cancelling $\triangle AB'C$.

Ex. 149. Prove Proposition IV, by means of Proposition VIII.

Ex. 150. What is the ratio of $\triangle ABC$ and $\triangle XYZ$ if $\angle s\ B$ and Y each equal $50°$; and $AB = 5$, $BC = 16$, $XY = 8$, and $YZ = 10$?

PROPOSITION IX. PROBLEM

342. *Construct a triangle equal to a given polygon.*



Given pentagon $ABCDE$.

Required to construct a $\triangle = ABCDE$.

I. *Change $ABCDE$ into an equal quadrilateral.*

Construction. 1. Draw diagonal AC, cutting off $\triangle ABC$.
2. Draw $BF \parallel AC$, meeting DC extended at F. Draw AF.

Statement. $AFDE = ABCDE$.

Proof:

STATEMENTS	REASONS
1. $\triangle ABC$ and $\triangle ACF$ have base AC and equal altitudes, the distance between $\parallel$s AC and BF.	1. Definition of altitude; § 129.
2. $\therefore \triangle ABC = \triangle ACF$.	2. Why?
3. $AFDE = ACDE + \triangle ACF$.	3. Why?
4. $ABCDE = ACDE + \triangle ABC$.	4. Why?
5. $\therefore AFDE = ABCDE$.	5. Why?

II. *Change $AFDE$ into an equal triangle.*

Construction. 1. Draw AD; draw $GE \parallel AD$, meeting FD extended at G; draw AG.

Statement. $\triangle AFG =$ quadrilateral $AFDE$.

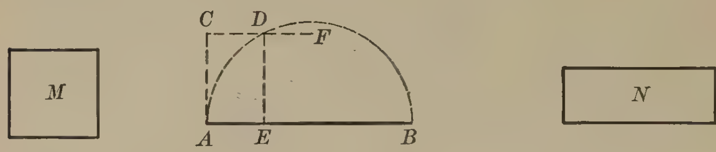
(Proof left to the pupil.)

Ex. 151. Draw a reasonably large quadrilateral. Construct a triangle equal to it. Then construct a square equal to the triangle.

Ex. 152. Construct a square equal to a given pentagon.

PROPOSITION X. PROBLEM

343. *Construct a rectangle equal to a given square, having the sum of its base and altitude equal to a given segment.*



Given square M and segment AB.

Required to construct a rectangle equal to M, having the sum of its base and altitude equal to AB.

Construction. 1. On AB as diameter construct semicircle ADB.

2. Construct $AC \perp AB$, making AC = side of M.

3. Construct $CF \parallel AB$, intersecting arc ADB at D.

4. Construct $DE \perp AB$.

5. Construct $\square N$, having its base $= BE$ and its altitude $= AE$.

Statement. Rectangle N = square M.

Proof:

	STATEMENTS	REASONS
1.	$AE : DE = DE : BE$.	1. § 290.
2.	$\therefore \overline{DE}^2 = AE \times BE$.	2. Why?
3.	$\therefore$ area of M = area of N.	3. Why?

Discussion. The construction is impossible when the side of the square is more than $\frac{1}{2}$ AB. Why?

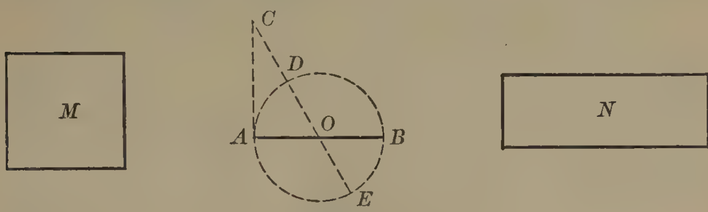
Note. § 343 suggests a geometrical solution of a quadratic of the form $x^2 - tx + m^2 = 0$.

From this equation, $m^2 = x(t - x)$. m corresponds to a side of the square, $x(t - x)$ corresponds to the area of the rectangle equal to the square, and t corresponds to the given segment, for $x + (t - x) = t$.

Solve $x^2 - 10\,x + 16 = 0$ geometrically and check the solution algebraically.

PROPOSITION XI. PROBLEM

344. *Construct a rectangle equal to a given square, having the difference between its base and altitude equal to a given segment.*



Given square M and segment AB.

Required to construct a rectangle equal to M, having the difference between its base and altitude equal to AB.

Construction. 1. On AB as diameter, construct $\odot ADB$.

2. Draw $AC \perp AB$, making AC = a side of M.

3. Through the center O, draw CO, intersecting the $\odot$ at D and E.

4. Construct $\square N$, having base CE and altitude CD.

Statement. $\square N$ is the required rectangle.

Proof:

	Statements	Reasons
1.	$CE - CD = DE$, or AB.	**1.** Why?
2.	$\therefore$ the base of N – the alt. of $N = AB$.	**2.** Ax. 2, § 49.
3.	AC is tangent to the circle.	**3.** Why?
4.	$\therefore CE \times CD = \overline{CA}^2$.	**4.** § 287.
5.	$\therefore$ area of N = area of M.	**5.** Why?

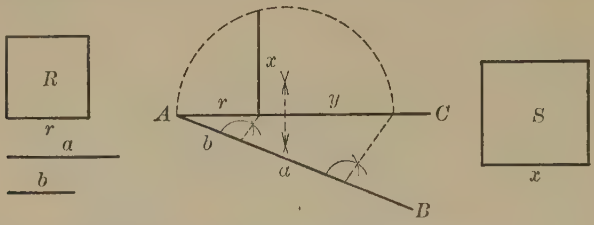
Discussion. The construction is always possible, since a secant can always be drawn through the center of the circle from an exterior point.

Note. § 344 suggests a geometrical solution of a quadratic of the form $x^2 + tx - m^2 = 0$, for this equation may be written, $m^2 = x(t + x)$.

Solve the equation $x^2 + 8x - 9 = 0$ geometrically.

PROPOSITION XII. PROBLEM

345. *Construct a square having a given ratio to a given square.*



Given square R and the segments a and b.

Required to construct a square S such that $S : R = a : b$.

Analysis. 1. Let x = one side of the required square.

2. $\therefore x^2 : r^2 = a : b.$

3. $\therefore bx^2 = ar^2$, or $x^2 = \left(\frac{ar}{b}\right) \times r.$ Algebra.

4. $\therefore r : x = x : \left(\frac{ar}{b}\right).$ Why?

5. $\therefore x$ is the mean proportional between r and $\frac{ar}{b}$.

6. Let $y = \frac{ar}{b}$, or $by = ar$.

7. $\therefore b : a = r : y.$ Why?

8. $\therefore y$ is the fourth proportional to b, a, and r. Def.

Construction. 1. Construct y as determined in Step 8.

2. Construct x as determined in Step 5.

3. Construct square S having x as side.

Statement. $S : R = a : b.$

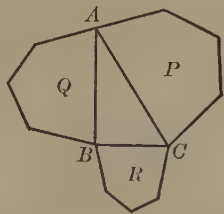
Proof:

STATEMENTS	REASONS
1. $S : R = x^2 : r^2.$	1. § 336.
2. But $x^2 = ry$, and $y = \frac{ar}{b}$.	2. Construction.

(Complete the proof by substituting in Step 1, for x^2, and, later, for y, their values given in Step 2.)

PROPOSITION XIII. THEOREM

346. *If similar polygons be constructed on the three sides of a right triangle as corresponding sides, the area of the polygon on the hypotenuse equals the sum of the areas of the polygons on the two legs.*



Hypothesis. $\triangle ABC$ is a rt. $\triangle$. Polygons P, Q, and R are similar polygons having AC, AB, and BC as corresponding sides.

Conclusion. Area of $P =$ area of $R +$ area of Q.

Proof:

STATEMENTS	REASONS
1. $Q : P = \overline{AB}^2 : \overline{AC}^2$.	1. § 336.
2. $R : P = \overline{BC}^2 : \overline{AC}^2$.	2. Why?
3. $\therefore \dfrac{Q}{P} + \dfrac{R}{P} = \dfrac{\overline{AB}^2}{\overline{AC}^2} + \dfrac{\overline{BC}^2}{\overline{AC}^2}$.	3. Ax. 3, § 49.
4. $\therefore \dfrac{Q+R}{P} = \dfrac{\overline{AB}^2 + \overline{BC}^2}{\overline{AC}^2}$.	4. Adding.
5. But $\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$.	5. Why?
6. $\therefore \dfrac{Q+R}{P} = \dfrac{\overline{AC}^2}{\overline{AC}^2}$.	6. Ax. 2, § 49.
7. $\therefore Q + R = P$.	7. Line 5, § 248.

Note. This theorem is, obviously, a general theorem which includes the theorem on page 216 as a special case.

Ex. 153. Construct an equilateral triangle equal to the sum of two given equilateral triangles.

Ex. 154. Construct an equilateral triangle equal to the difference of two given equilateral triangles.

Miscellaneous Exercises

Ex. 155. Construct two similar irregular quadrilaterals. Then construct a third quadrilateral, similar to them, and equal to their sum.

Ex. 156. (*a*) Make a reasonably large pentagon, and construct a triangle equal to the pentagon. (*b*) Measure the base and altitude of the triangle, and compute the area of the triangle. (*c*) What is the area of the pentagon?

Ex. 157. The area and perimeter of one triangle are 96 sq. in. and 48 in. respectively. The perimeter of a similar triangle is 75 in. What is the area of the second triangle?

Ex. 158. Construct a quadrilateral similar to a given quadrilateral and equal to four times it in area.

Ex. 159. Construct a pentagon similar to a given pentagon and equal to one half of it.

Ex. 160. Construct a rectangle equal to a given square and having the sum of its base and altitude equal to 3 in.

Ex. 161. Construct a rectangle equal to a given square and having the difference of its base and altitude equal to 2 in.

Ex. 162. Construct a square equal to twice the area of a given quadrilateral.

Ex. 163. Construct a square which shall be 5 times as large as a given square.

Ex. 164. Construct a square which shall be three fourths as large as a given square.

Note. These exercises and the foregoing propositions make it clear that a polygon can be constructed by ruler and compass alone which has any given ratio to a given polygon, and which is similar to it, or which may have a different but, to a certain extent, a specified shape and have, even, specified parts.

On the other hand, one of the famous problems of construction which cannot be solved by ruler and compass alone is that of making a square equal to a given circle.

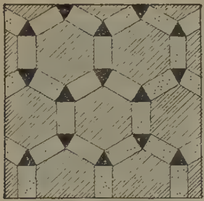
BOOK V

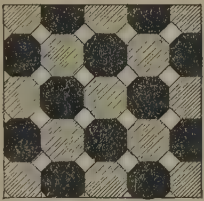
REGULAR POLYGONS. MEASUREMENT OF THE CIRCLE

347. Review the definitions given in § 120, § 122, § 123, and § 177. Also, recall § 145.

348. A **regular polygon** is a polygon which is both equilateral and equiangular.

The figures below illustrate some uses of regular polygons:





TWO LINOLEUM PATTERNS

Ex. 1. Prove that the exterior angles at the vertices of a regular polygon are equal.

Ex. 2. What is the perimeter of a regular pentagon one of whose sides is 7 in.? of a regular octagon one of whose sides is 6 in.?

Ex. 3. In § 145, we have proved that the sum of the angles of any polygon having n sides is $(n - 2)$ st. $\measuredangle$.

How large is each angle of a regular polygon having: (a) 3 sides? (b) 4 sides? (c) 5 sides? (d) 6 sides? (e) 8 sides? (f) 10 sides?

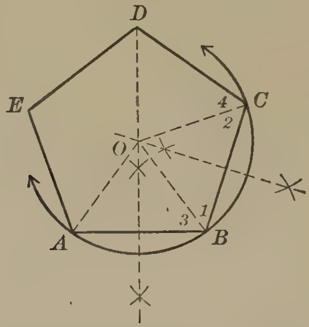
Ex. 4. (a) Four square tile can be used to cover the space around a point. (Why?)

(b) In the shape of what other regular polygon can tile be made in order that the surface around a point can be completely covered by using tile of the same shape?

349. *Each angle of a regular polygon having n sides is* $\left(\frac{n-2}{n}\right)$ *st.* $\measuredangle$. (See Ex. 3.)

* PROPOSITION I. THEOREM

350. *A circle can be circumscribed about any regular polygon.*



Hypothesis. $ABCDE$ is a regular polygon.

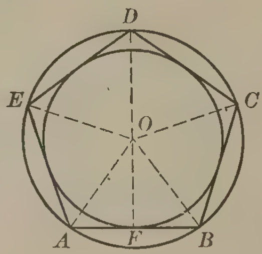
Conclusion. A circle can be circumscribed about $ABCDE$.

Plan. Prove the ⊙ through A, B, and C passes through D and E.

Proof:

Statements	Reasons
1. A ⊙ can be drawn through A, B, and C. Let O be its center, and OA, OB, and OC radii of it.	1. § 173; § 174.
2. $\therefore \triangle OAB \cong \triangle ODC$.	2. Give the proof.
3. $\therefore OA = OD$.	3. Why?
4. $\therefore$ the ⊙ABC goes through D.	4. § 172, (*b*).

5. Similarly this circle can be proved to pass through point E; hence it circumscribes $ABCDE$.



351. Cor. 1. *A circle can be inscribed in any regular polygon.*

Proof. 1. AB, BC, CD, etc., are equidistant from center O of the ⊙$ABCDE$.

2. If OF is the distance from O to AB, the ⊙ with center O and radius OF is tangent to AB, BC, CD, etc.

352. The **center** of a regular polygon is the common center of the circumscribed and inscribed circles; as O.

The **radius** of a regular polygon is the distance from its center to any vertex; as OA.

The **apothem** of a regular polygon is the distance from its center to any side; as OF.

The **central angle** of a regular polygon is the angle between the radii drawn to the ends of any side; as $\angle AOB$.

The **vertex angle** of a regular polygon is the angle between two sides of the polygon.

353. Cor. 2. *The central angle of a regular n-gon is* $\dfrac{360°}{n}$.

Ex. 5. Find the number of degrees in the central angle and in the vertex angle of a regular polygon of: (*a*) 3 sides; (*b*) 4 sides; (*c*) 5 sides; (*d*) 6 sides; (*e*) 8 sides; (*f*) 10 sides.

Ex. 6. Find also the sum of the central angle and the vertex angle in each part of Ex. 5. Do the results suggest any theorem?

Ex. 7. Prove that any radius of a regular polygon bisects the angle to whose vertex it is drawn.

Ex. 8. In regular polygon $ABCDE$, prove that diagonals AC, BD, CE, EB, and DA are equal.

Ex. 9. Prove that the apothem of a regular polygon bisects the side to which it is drawn.

Suggestion. Draw the circumscribed circle of the polygon.

Ex. 10. In regular polygon $ABCDE$, prove that the diagonals AC and AD divide $\angle BAE$ into three equal parts.

Suggestion. Draw the circumscribed circle.

Ex. 11. In regular polygon $ABCDEF$, prove that the diagonals AC, AD, and AE divide $\angle BAF$ into four equal parts.

Ex. 12. In any regular polygon of which A, B, and C are three consecutive vertices, and of which O is the center, prove that OB is the perpendicular bisector of AC.

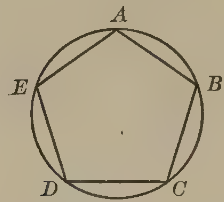
Ex. 13. Prove that the points of contact of the circle inscribed in a regular polygon divide the circle into as many equal parts as the polygon has sides.

Suggestion. Prove it for the case of a regular pentagon. Draw the chords connecting the points of contact, taken in order.

Note. Additional Exercises 1 to 7, p. 305, can be studied now.

PROPOSITION II. THEOREM

354. *If a circle be divided into any number of equal arcs, the chords of these arcs form a regular inscribed polygon of that number of sides.*



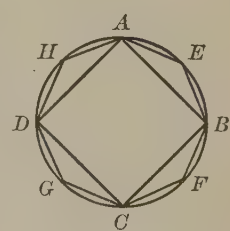
Hypothesis. $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD} = \overset{\frown}{DE} = \overset{\frown}{EA}$ in $\odot O$.

Conclusion. $ABCDE$ is a regular pentagon.

Plan. Prove $ABCDE$ equilateral and equiangular.

Proof: Statements	Reasons
1. $AB = BC = CD = DE$, etc.	1. Why?
2. $\angle A$ i. m. b. $\frac{1}{2}\overset{\frown}{BCDE}$; $\angle B$ i. m. b. $\frac{1}{2}\overset{\frown}{CDEA}$; $\angle C$ i. m. b. $\frac{1}{2}\overset{\frown}{DEAB}$; etc.	2. Why? (For "i. m. b." see § 210.)
3. $\overset{\frown}{BCDE} = \overset{\frown}{CDEA} = \overset{\frown}{DEAB}$, etc.	3. Why?
4. $\therefore \angle A = \angle B = \angle C$, etc.	4. Why?
5. $\therefore ABCDE$ is a regular polygon.	5. Why?

355. Cor. 1. *If from the mid-point of each arc subtended by a side of a regular polygon lines be drawn to its extremities, a regular inscribed polygon of double the number of sides is formed.*

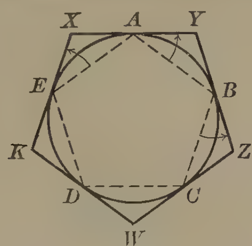


356. Cor. 2. *An equilateral polygon inscribed in a circle is regular.*

Ex. 14. A circle of radius 10 in. is divided into six equal arcs. How long is each side of the regular polygon formed by joining the points of division? How long is the apothem of this polygon? How large is the central angle? How large is the vertex angle?

PROPOSITION III. THEOREM

357. *If a circle be divided into any number of equal arcs, the tangents at the points of division form a regular circumscribed polygon of that number of sides.*



Hypothesis. $\odot$ ACD is divided into five equal arcs, $\widehat{AB}$, $\widehat{BC}$, etc. XY, ZY, etc. are tangent to $\odot$ ACD at A, B, etc., forming pentagon $XYZWK$.

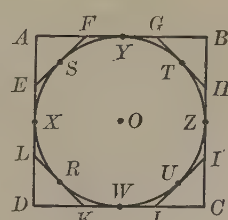
Conclusion. $XYZWK$ is a regular pentagon.

Plan. Prove $XYZWK$ equilateral and equiangular.

Proof:

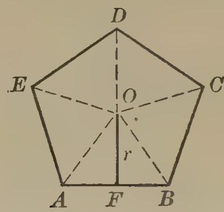
STATEMENTS	REASONS
1. Draw AB, BC, CD, DE, and EA.	1. Why possible?
2. $\triangle AXE$, AYB, BCZ, etc., are cong. isos. $\triangle$.	2. Give the proof.
3. $\therefore AY = YB = BZ = ZC$, etc.	3. Why?
4. $\therefore YZ = ZW = WK$, etc.	4. Why?
5. $\angle Y = \angle Z = \angle W$, etc.	5. Give the proof.
6. $\therefore XYZWK$ is a regular polygon.	6. Why?

358. Cor. *Tangents drawn to the circle at the mid-points of the arcs included between two consecutive points of contact of a regular circumscribed polygon form, with the sides of the original circumscribed polygon, a regular circumscribed polygon having double the number of sides.*



*PROPOSITION IV. THEOREM

359. *The area of a regular polygon is equal to one half the product of its apothem and its perimeter.*



Hypothesis. The perimeter of regular polygon AC is p and the apothem OF is r.

Conclusion. Area of $ABCDE = \frac{1}{2} rp$.

Proof:

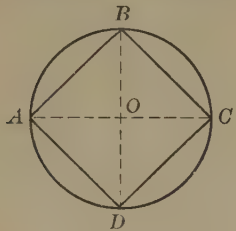
STATEMENTS	REASONS
1. Draw the radii OA, OB, OC, etc.	1. Why possible?
2. r is the common altitude of the $\triangle OAB, OBC, OCD$, etc.	2. Definition of apothem.
3. $\triangle OAB = \frac{1}{2} r \times AB$; $\triangle OBC = \frac{1}{2} r \times BC$; etc.	3. Why?
4. $\therefore \triangle OAB + \triangle OBC + \triangle OCD$, etc. $= \frac{1}{2} r \times AB + \frac{1}{2} r \times BC + \frac{1}{2} r \times CD$, etc.	4. Why?
5. $\therefore$ area $ABCDE = \frac{1}{2} r(AB + BC + CD$, etc.$)$.	5. Factoring.
6. $\therefore$ area $ABCDE = \frac{1}{2} rp$.	6. Ax. 7, § 49.

Ex. 15. What is the area of the inscribed regular polygon obtained in Exercise 14, page 236?

360. Construction of regular polygons. In *practical geometry*, a regular polygon of any number of sides can be constructed (at least approximately) by drawing at the center of the circle a central angle of the proper size by means of the protractor. *In theoretical geometry*, however, only the straight-edge and compass can be used for constructions.

PROPOSITION V. PROBLEM

361. *Inscribe a square in a given circle.*



Given circle O.

Required to inscribe a square in circle O.

Construction. 1. Draw AC and BD, perpendicular diameters.

2. Draw chords AB, BC, CD, and AD.

Statement. $ABCD$ is the required square.

(Proof left to the pupil. Use § 354.)

362. Cor. *By bisecting the arcs subtended by the sides of a square, the circle is separated into eight equal arcs, and a regular inscribed octagon can be constructed. Similarly regular polygons of* 16, 32, *etc. sides can be constructed.*

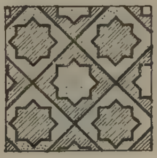
Ex. **16.** Construct a square whose diagonal is 4 in.

Ex. **17.** Compute, correct to one decimal place, the length of the side, the apothem, and the area of the square obtained in Ex. 16.

Ex. **18.** Construct a circle of radius 1.5 in.

Ex. **19.** Circumscribe a square about, and inscribe a regular octagon in, a circle.

Ex. **20.** Derive formulas for the side, perimeter, apothem, and area of a square inscribed in a circle of radius r.



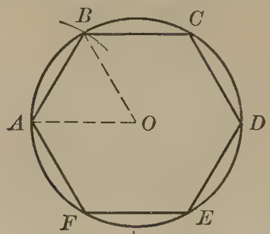
Ex. **21.** Construct within a circle having a 3-in. radius an eight-pointed star like the one which forms the central unit of the adjoining linoleum pattern.

Ex. **22.** A designer wishes to make a pattern for the octagonal top of a taboret whose longest diagonal is to be 18 in. Make a scale drawing of the octagon, letting 1 in. represent 3 in.

Note. Additional Exercises 8 to 17, p. 305, can be studied now.

PROPOSITION VI. PROBLEM

363. *Inscribe a regular hexagon in a circle.*



Given circle O.

Required to inscribe a regular hexagon in circle O.

Analysis. The central angle of a regular hexagon is 60°.

Construction. Draw any radius OA. With A as center and OA as radius draw an arc cutting the ⊙ at B.

Statement. $\overset{\frown}{AB} = \frac{1}{6}$ of the ⊙, and may be applied 6 times to the circle. The chords of these arcs form the regular inscribed hexagon, $ABCDEF$.

Proof:

STATEMENTS	REASONS
1. Draw OB. $\triangle AOB$ is an equilateral $\triangle$.	1. Give the proof.
2. $\therefore \overset{\frown}{AB} = \frac{1}{6}$ of the circle.	2. Give the proof.
3. $\therefore ABCDEF$ is a regular inscribed hexagon.	3. Why?

364. Cor. 1. *Chords joining the alternate vertices of a regular inscribed hexagon, starting with any vertex, form a regular inscribed triangle.*

365. Cor. 2. *Regular inscribed polygons of 12, 24, 48, etc., sides can be constructed.* (§ 355.)

Ex. 23. Construct a regular hexagon whose side is 1 in.

Ex. 24. Prove that the diagonals joining alternate vertices of a regular hexagon are equal.

Ex. 25. Prove that the radii of a regular inscribed hexagon divide it into six congruent equilateral triangles.

Ex. 26. Prove that diagonals AD, BE, and CF of a regular hexagon $ABCDEF$ are diameters of its circumscribed circle.

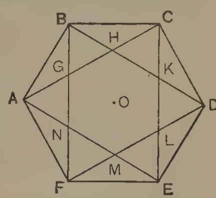
Ex. 27. Prove that the opposite sides of a regular hexagon are parallel.

Ex. 28. Prove that diagonal FC of regular hexagon $ABCDEF$ is parallel to sides AB and DE.

Ex. 29. Prove that diagonal AD of regular hexagon $ABCDEF$ is perpendicular to diagonal BF and bisects it.

Ex. 30. Prove that diagonal BF is parallel to diagonal CE.

Ex. 31. Prove that quadrilateral $ACDF$, formed by diagonals AC and DF and sides AF and CD of regular hexagon $ABCDEF$, is a rectangle.



Ex. 32. In the adjoining figure $ABCDEF$ is a regular hexagon, with certain of its diagonals drawn. Prove $\triangle BHC \cong \triangle CKD \cong \triangle DLE$, etc.

Ex. 33. In the same figure, prove that $\triangle$s CHK, DKL, LME, etc., are congruent.

Ex. 34. In the same figure, prove that $GHKLMN$ is a regular hexagon.

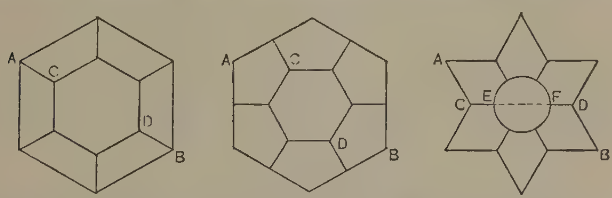
Ex. 35. Compute, correct to one decimal place, the side, the perimeter, the apothem, and the area of the regular hexagon inscribed in a circle of radius 10 in.

Ex. 36. Compute also the side, perimeter, apothem, and area of the regular triangle inscribed in the circle of radius 10 in.

Ex. 37. Prove that the area of the inscribed equilateral triangle is one half the area of the inscribed regular hexagon, in the same circle.

Ex. 38. Find the area of a regular hexagon whose apothem is 6 in.

Ex. 39. Construct one of the following designs in a 2-in. circle:

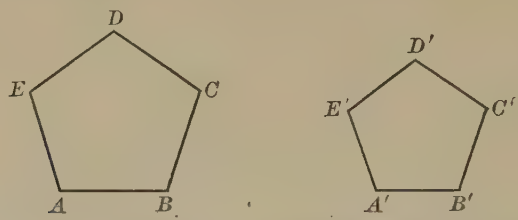


Note 1. Optional Topic D, page 265, might well be done now, if the class has studied numerical trigonometry.

Note 2. Additional Exercises 18 to 33, page 306, can be studied now.

PROPOSITION VII. THEOREM

366. *Regular polygons of the same number of sides are similar.*



Hypothesis. $ABCDE$ and $A'B'C'D'E'$ are regular polygons of 5 sides.

Conclusion. $ABCDE \sim A'B'C'D'E'$.

Plan. Prove the corres. ∡ equal, and corres. sides proportional.

Proof:

STATEMENTS	REASONS
1. Each ∠ of both polygons $= \left(\frac{5-3}{5}\right)180°$.	1. § 349.
2. ∴ the polygons are mutually equiangular.	2. Definition.
3. $AB = BC = CD = DE = EA$. $A'B' = B'C' = C'D' = D'E' = E'A'$.	3. Why?
4. $\therefore \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}$.	4. Ax. 6, § 49.
5. $\therefore ABCDE \sim A'B'C'D'E'$.	5. Why?

Note. A similar proof is given when the polygons have n sides.

1. Each angle of the two polygons $= \left(\frac{n-2}{n}\right)180°$, by § 349.
2. ∴ the polygons are mutually equiangular.
3. Since $AB = BC = CD$, etc. and $A'B' = B'C' = C'D'$, etc.,

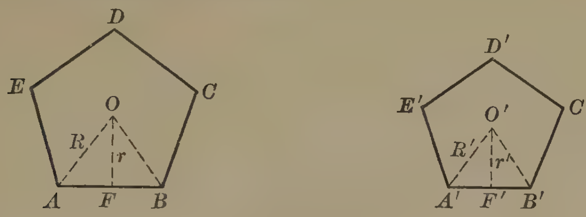
then $$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}, \text{ etc.}$$

4. Therefore the polygons are similar, by definition.

Ex. 40. In the figure for § 357, prove that the radius drawn to the vertex Y is the perpendicular bisector of the side AB of the inscribed polygon.

PROPOSITION VIII. THEOREM

367. *The perimeters of two regular polygons of the same number of sides have the same ratio as their radii, or as their apothems.*



Hypothesis. P and P' are the perimeters; O and O' are the centers; R and R' are the radii; and r and r' are the apothems respectively of the regular polygons AC and $A'C'$ of the same number of sides.

Conclusion. $\dfrac{P}{P'} = \dfrac{R}{R'} = \dfrac{r}{r'}$.

Proof:

STATEMENTS	REASONS
1. Let O and O' be the centers of the polygons AC and $A'C'$; let OA, OB, $O'A'$, and $O'B'$ be radii; and OF and $O'F'$ be apothems.	1. § 352.
2. Polygon $AC \sim$ polygon $A'C'$.	2. § 366.
3. $\therefore \dfrac{P}{P'} = \dfrac{AB}{A'B'}$.	3. § 297.
4. $OA = OB$; and $O'A' = O'B'$.	4. Why?
5. $\therefore \dfrac{OA}{O'A'} = \dfrac{OB}{O'B'}$.	5. Ax. 6, § 49.
6. $\angle AOB = \angle A'O'B'$.	6. § 353.
7. $\therefore \triangle AOB \sim \triangle A'O'B'$.	7. § 280.
8. $\therefore \dfrac{AB}{A'B'} = \dfrac{R}{R'} = \dfrac{r}{r'}$.	8. § 282.
9. $\therefore \dfrac{P}{P'} = \dfrac{R}{R'} = \dfrac{r}{r'}$.	9. Ax. 1, § 49.

368. Cor. *The areas of two regular polygons of the same number of sides have the same ratio as the squares of their radii or as the squares of their apothems.*

Proof. 1. Let K and K' represent the areas of the polygons AC and $A'C'$ respectively.

Then $$\frac{K}{K'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}. \qquad \S\ 336$$

2. $$\frac{K}{K'} = \frac{R^2}{R'^2} = \frac{r^2}{r'^2}. \qquad \text{See Step 8, } \S\ 367$$

Ex. 41. Compare the central angle of a regular hexagon with the exterior angle at any vertex. (See § 349 and § 353.)

Ex. 42. The perimeters of regular inscribed polygons of 6 and 12 sides respectively inscribed in a circle of diameter 2 are approximately 6 in. and 6.21 in. respectively. What are the perimeters of regular inscribed polygons of 6 and 12 sides respectively in a circle of diameter 4? of diameter 7? of diameter 10?

Ex. 43. The perimeters of regular polygons of 4 and 8 sides respectively circumscribed about a circle of diameter 2 in. are 8 in. and 6.63 in. respectively. What are the perimeters of regular circumscribed polygons of 4 and 8 sides respectively circumscribed about a circle of diameter 6? of diameter 5?

Ex. 44. The area of a regular hexagon inscribed in a circle of radius 3 in. is 23.38 sq. in. What is the area of a regular hexagon inscribed in a circle of radius 6 in.? of one in a circle of radius 1 in.?

Ex. 45. The area of a regular octagon circumscribed about a circle of radius 1 in. is 3.3137 sq. in. What is the area of a regular octagon circumscribed about a circle of radius 2 in.? of radius 5 in.?

Ex. 46. Prove that the perimeter of any regular inscribed polygon is less than the perimeter of the regular circumscribed polygon of the same number of sides.

Ex. 47. In any circle, inscribe a square, a regular octagon, and a regular 16-gon. Prove that the perimeter of the square is less than that of the octagon, and that the latter is less than the perimeter of the 16-gon.

Ex. 48. About a large circle, circumscribe a square, a regular octagon, and a regular 16-gon. Prove that the perimeter of the square is greater than that of the octagon, and that the latter is greater than the perimeter of the 16-gon.

MENSURATION OF THE CIRCLE

369. Before taking up the mensuration of the circle, we shall define certain ideas, fundamental in mathematics.

370. Variable, constant, and limit of a variable.

Example 1. Consider the numbers 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc.

If x is a literal number which has these values successively, then $x - 0$ ultimately becomes less than any small positive number like $\frac{1}{1000}$ or $\frac{1}{1000000}$, and, thereafter, $x - 0$ remains less than that number. We say that x approaches 0 as limit.

Example 2. Consider the numbers 1, $1\frac{1}{2}$, $1\frac{3}{4}$, $1\frac{7}{8}$, etc.

If x is a literal number which has these values successively, then ultimately $2 - x$ becomes less than any small positive number, like $\frac{1}{100000}$, and thereafter remains less than that number. We say that x approaches 2 as limit.

(*a*) A **variable** is a number which assumes different values during a particular discussion; as x in Examples 1 and 2, above.

(*b*) A **constant** is a number which has a fixed value during a particular discussion; as 0 in Example 1, and 2 in Example 2.

(*c*) The **limit of a variable** is a constant such that the numerical value of the difference between it and the variable becomes and remains less than any small positive number.

The symbol for "approaches the limit" is $\rightarrow$.

371. Two theorems about limits.

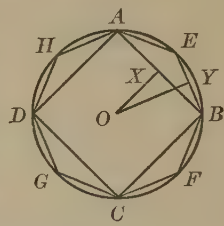
(*a*) *If a variable x approaches a limit l, then cx, where c is a constant, approaches the limit cl.*

$cl - cx = c(l - x)$. Since $x \rightarrow l$, then $l - x$ becomes and remains less than any small positive number. $\therefore c(l - x)$ or $cl - cx$ also becomes and remains less than any small positive number.

(*b*) *If two variables are constantly equal, and each approaches a limit, their limits are equal.*

If the limits were unequal, the variables could no longer be equal when they came to be near their limits.

372. Length and area of a circle. We cannot define the length of a circle as the ratio of the circle to a unit of linear measure, because we cannot lay off a straight line segment on a circle. A like difficulty prevents defining the area of a circle as the ratio of the interior of the circle to the customary unit of surface measure. We proceed as follows:



(*a*) In the adjoining circle are inscribed a square and a regular octagon. Imagine also that the regular polygons of 16, 32, etc. sides are inscribed.

It is clear that these polygons come closer and closer to the circle. If we find the perimeters and areas of these polygons, they become larger as the number of their sides increases and will be closer and closer approximations to the length and area of the circle.

We therefore define the length and area of a circle as follows:

(*b*) **The length of a circle** is the limit which the perimeters of regular inscribed polygons approach when the number of their sides is increased indefinitely.

(*c*) **The area of a circle** is the limit which the areas of regular inscribed polygons approach when the number of their sides is increased indefinitely.

(*d*) It is proved in more advanced courses in geometry that the perimeters and areas of regular *circumscribed polygons* of a circle also approach respectively the length and area of the circle when the number of their sides is increased indefinitely.

373. Observing apothems OX, OY, etc. above, the following theorem, which is proved in more advanced courses, is clear:

If regular polygons be inscribed in a circle, the apothems of the polygons approach the radius of the circle as limit as the number of the sides of the polygons increases indefinitely.

PROPOSITION IX. THEOREM

374. *The circumferences of two circles have the same ratio as their radii or their diameters.*

Hypothesis. C_1 and C_2 are the circumferences of two circles whose radii are r_1 and r_2 and whose diameters are d_1 and d_2.

Conclusion. $\frac{C_1}{C_2} = \frac{r_1}{r_2} = \frac{d_1}{d_2}.$

Proof:

STATEMENTS	REASONS
1. Inscribe in the ⓢ regular polygons having the same number of sides. Let p_1 and p_2 be their perimeters.	1. A possible construction.
2. $\therefore p_1 : p_2 = r_1 : r_2.$	2. § 367.
3. $\therefore p_1 \times r_2 = p_2 \times r_1.$	3. Why?
4. Repeatedly double the number of sides of the two polygons, keeping the no. the same in the two circles. Then $p_1 \rightarrow C_1$ and $p_2 \rightarrow C_2$.	4. § 372, (*b*).
5. $\therefore p_1 \times r_2 \rightarrow C_1 \times r_2$; and $p_2 \times r_1 \rightarrow C_2 \times r_1$.	5. § 371, (*a*).
6. $\therefore C_1 \times r_2 = C_2 \times r_1.$	6. § 371, (*b*).
7. $\therefore C_1 : C_2 = r_1 : r_2.$	7. § 252.
8. $\therefore C_1 : C_2 = d_1 : d_2.$	8. Why?

375. Cor. $C_1 : d_1 = C_2 : d_2.$ § 255

That is, *the ratio of the circumference of one circle to its diameter equals the ratio of the circumference of any other circle to its diameter.*

376. The constant ratio of a circumference of a circle to its diameter is denoted by the Greek letter π.

377. Formulas for the circumference of a circle. Since $\frac{C}{d} = \pi$ for any circle, then $C = \pi d$.

Since $d = 2r$, then $C = 2\pi r$.

Only approximate values of π can be given. Two of the most useful approximations are: $\pi = 3.1416$, and $\pi = 3\frac{1}{7}$.

Note. The determination of the value of π and of what sort of number π is has been one of the most famous problems of mathematics.

The Egyptians recognized that $C \div d$ is constant, and obtained for this ratio a value which corresponds to 3.1605. The Babylonians and Hebrews were content with the much less accurate value, $\pi = 3$.

The method employed in this text was introduced by Antiphon (469–399 B.C.), improved by Bryson (a contemporary, probably), and finally carried out arithmetically in a remarkable manner by Archimedes (287–212 B.C.) in a pamphlet on the mensuration of the circle. Antiphon suggested the use of inscribed regular polygons of 4, 8, etc., sides as a means of approximating the length of the circle, and Bryson suggested using at the same time the corresponding circumscribed regular polygons. Archimedes employed inscribed and circumscribed regular polygons having 3, 6, $\cdots$ 96 sides in his computation, and showed that $\pi > 3\frac{10}{71}$ and $< 3\frac{1}{7}$. The methods employed by Archimedes remained for a long time the standard procedure in efforts to compute π. As mathematical skill increased, formulas for π were derived, particularly in trigonometric form, which enabled diligent computers to obtain the value to more and more decimal places. Vieta (1540–1603) gave for π the value 3.141529653. Others carried out the computation to as many as 700 decimal places.

A Holland mathematician, Huygens (1629–1695), at the age of twenty-five, proved some theorems which made it possible to obtain from a regular hexagon as accurate a value for π as Archimedes obtained from the regular 96-gon.

Mathematicians were particularly interested in determining what kind of number π is. In 1766–1767, Lambert proved that it is not rational; *i.e.*, that it cannot be expressed as the quotient of two integers. In 1882, through methods introduced by Hermite in 1873, Lindeman proved that π is a transcendental number; *i.e.*, that it cannot be the root of an ordinary algebraic equation. This was the goal toward which previous efforts had been directed, and thus completely solved a problem to which many of the great mathematicians had given some attention.

Ex. 49. Find the circumference of a circle whose diameter is 5 in.; one whose diameter is 8 in.; one whose diameter is 15 in.

Note. Obtain the result correct to one decimal place.

Ex. 50. How long must the rubber for the tire of a wheel be if the diameter of the wheel is 12 in.?

Ex. 51. What is the circumference of a circle whose radius is 3 ft.? One whose radius is 25 in.?

Ex. 52. If the diameter of a circle is 24 in., how long is an arc of 60°? How long is an arc of 72°? How long is an arc of 80°?

Ex. 53. (*a*) What is the diameter of a circular running track whose length along its inside edge is one fourth of a mile?

Suggestion. One mile = 5280 ft.

(*b*) If the track is 25 ft. wide, determine the distance around it on its outside edge.

Ex. 54. A fly wheel in an engine room has a diameter of 10 ft. The wheel revolves 200 times per minute. Through how many feet does a point on the outside of it pass in a minute?

Ex. 55. The *rim speed* of a wheel is the distance through which a point on the rim passes in a minute. A good average speed for an emery stone is about 5000 ft. per minute.

(*a*) What is the rim speed of a 9-in. emery wheel which is revolving 2100 times per minute?

(*b*) How fast may an emery wheel be revolved in order that its rim speed shall not be more than 5000 ft. per minute, if the diameter of the wheel is 6 in.?

Ex. 56. (*a*) A 32×4 tire is one which is 4 in. in diameter, and which has an outside diameter of 32 in. What is the inside diameter?

(*b*) Why is a 33×4.5 tire a proper oversize tire for a wheel equipped with a 32×4 tire?

(*c*) What is a proper oversize tire for a wheel equipped with $30 \times 3\frac{1}{2}$ tires?

Ex. 57. (*a*) How far forward is a car carried by rear wheels equipped with 32×4 tires when they revolve once?

(*b*) Repeat (*a*) when the tires are 33×4.5 tires.

Ex. 58. Draw any circle. Construct the circle:

(*a*) whose circumference is 3 times that of the given circle.

(*b*) whose circumference is $\frac{1}{2}$ that of the given circle.

(*c*) whose circumference is twice that of the given circle.

Note. Additional Exercises 34 to 45, page 307, can be studied now.

PROPOSITION X. THEOREM

378. *The area of a circle is the product of one half its radius and its circumference.*

Hypothesis. r is the radius, C is the circumference, and S is the area of the circle.

Conclusion. $S = \frac{1}{2} r \times C.$

Proof:

STATEMENTS	REASONS
1. Circumscribe a regular polygon about the circle, having perimeter P and area K.	1. A possible construction.
2. $\therefore K = \frac{1}{2} r \times P.$	2. § 359.
3. Let the number of sides be increased indefinitely; then $K \rightarrow S$.	3. § 372, (*d*).
4. Also $P \rightarrow C$.	4. § 372, (*d*).
5. $\therefore \frac{1}{2} r \times P \rightarrow \frac{1}{2} r \times C.$	5. § 371, (*a*).
6. $\therefore S = \frac{1}{2} r \times C.$	6. § 371, (*b*).

379. Cor. 1. Since $C = 2\pi r$, then $S = \pi r^2$.

380. Cor. 2. Since $d = \frac{1}{2} r$, then $S = \frac{1}{4}\pi d^2$.

381. Cor. 3. *The areas of two circles have the same ratio as the squares of their radii, or the squares of their diameters.*

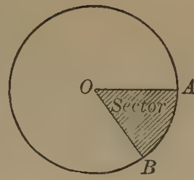
If S_1 and S_2 are the areas of the circles having radii r_1 and r_2, and diameters d_1 and d_2 respectively, then:

(*a*) $\dfrac{S_1}{S_2} = \dfrac{\pi r_1^2}{\pi r_2^2} = \dfrac{r_1^2}{r_2^2}.$ (*b*) Also $\dfrac{S_1}{S_2} = \dfrac{\frac{1}{4}\pi d_1^2}{\frac{1}{4}\pi d_2^2} = \dfrac{d_1^2}{d_2^2}.$

Ex. 59. Find the area of the circle whose radius is 8 in.

Ex. 60. Find the area of the circle whose diameter is 12 in.

382. **A sector of a circle** is the portion of the interior of a circle which is within a given central angle. The central angle is called the *angle of the sector.*



383. *The area of a sector is to the area of the circle as its angle is to 360°.*

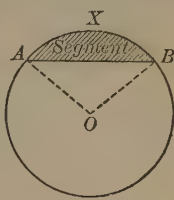
Example. What is the area of a sector whose angle is 45° in a circle whose radius is 10 in.?

Solution. 1. The area of the circle = 3.1416×10^2, or 314.16 sq. in.

2. If x = the area of the sector, then $x : 314.16 = 45 : 360$.

3. $\therefore x : 314.16 = 1 : 8$, or $x = 39.27$ sq. in.

384. **A segment of a circle** is that portion of the interior of a circle which is between a chord of the circle and its subtended arc; as segment AXB, indicated by the shaded part of the adjoining figure.



The area of a segment AXB may be determined by subtracting the area of $\triangle AOB$ from the area of sector $OAXB$.

Example. What is the area of the segment of a circle of radius 10 in. which is between a chord 10 in. long and its subtended arc?

Solution. 1. If radii are drawn to the ends of the chord, they form with the chord an equilateral triangle whose side is 10 in.

$$\text{The area of this triangle} = \frac{10^2}{4}\sqrt{3}, \text{ or } 43.3 \text{ sq. in.}$$

2. The sector included by these radii has an angle of 60°, and its area, therefore, is one sixth of the area of the circle.

The area of the circle = 3.1416×10^2, or 314.16 sq. in.

Therefore the area of the sector = $\frac{1}{6} \times 314.16$, or 52.36 sq. in.

3. The area of the segment = area of sector − area of triangle,
= 52.36 − 43.30, or 9.06 sq. in.

Ex. 61. Find the radius and area of the circle whose circumference is: (*a*) 26π in.; (*b*) 18π in.; (*c*) 15π in.

Ex. 62. The diameters of two circles are 6 and 8 respectively.

(*a*) What is the ratio of their areas?

(*b*) What is the ratio of their circumferences?

Ex. 63. The radii of three circles are 3, 4, and 12, respectively. What is the radius of a circle whose area equals the sum of the areas of these three circles?

Ex. 64. Find the area of the segment between a side of a regular inscribed hexagon and its minor arc if the radius of the circle is 12 in.

Ex. 65. If the radius of a circle is 4, what is the area of the segment between a side of the inscribed equilateral triangle and its minor arc?

Ex. 66. (*a*) Construct the circle whose area is four times that of a given circle.

(*b*) Construct the circle whose area is $\frac{1}{4}$ that of the given circle.

Ex. 67. Two pulleys in a machine shop are connected by a belt. One has a radius of 10 in. and the other a radius of 2 in. For each revolution of the large pulley how many revolutions will the small pulley make?

Ex. 68. What is the area of the ring between two concentric circles whose radii are 8 in. and 10 in. respectively?

Ex. 69. A circular grass plot, 100 ft. in diameter, is surrounded by a walk 4 ft. wide. Find the area of the walk.

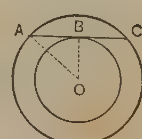
Ex. 70. How many tulip bulbs will be required for a circular flower bed 6 feet in diameter, allowing 16 sq. in. to each bulb?

Ex. 71. In a steam engine having a piston 20 in. in diameter, the pressure upon the piston is 90 lb. to the square inch. What is the total pressure upon the piston?

Ex. 72. A woman had a number of potted plants with which to plant a circular flower bed. She planned to make the bed 4 feet in diameter and found that she used up in that way just one half of her plants. Approximately how large should she make the bed to use up all of her plants?

Ex. 73. Prove that the area of the ring included between two concentric circles is equal to the area of a circle whose diameter is that chord of the outer circle which is tangent to the inner.

(Prove area of ring $= \frac{1}{4}\pi AC^2$.)



Ex. 74. Prove that the area of a circle is equal to four times the area of the circle described upon its radius as a diameter.

Ex. 75. What is the area of the circular ring between the circles inscribed in and circumscribed about a square whose side is 12 in.?

Ex. 76. In a circle of radius 12 in., how long is: (*a*) the arc of 72°? (*b*) the arc of 60°? (*c*) the arc of 45°?

Ex. 77. Two tangents to a circle meet at an angle of 36°. If the radius of the circle is 6 in., how long is the arc between the points of contact?

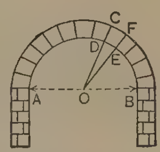
Ex. 78. In a circle of radius 5 in., what is the area of the sector whose central angle is 18°?

Ex. 79. In a circle of radius 10 in., what is the area of the smaller segment whose chord is:
(*a*) the side of an inscribed square?
(*b*) the side of an inscribed regular hexagon?
(*c*) the side of an inscribed regular triangle?

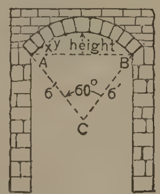
Ex. 80. How much greater is the cross section area of a hot air pipe 12 in. in diameter than one which is 9 in. in diameter?

Ex. 81. In erecting a hot air furnace for dwellings, certain pipes are installed for carrying the air to the various rooms of the house, and one or more other pipes are put in to convey cold air to the furnace. The cross section area of the cold air supply pipes must equal approximately the sum of the cross section areas of the warm air pipes.

A house is to have four warm air pipes 9 in. in diameter, and three 12 in. in diameter. One cold air pipe is to be installed. How large, approximately, must its diameter be?



Ex. 82. In the adjoining *semicircular arch* constructed about center O, the distance AB is 10 ft. If the arch is to be constructed of 13 stones of equal size, how long is each of the arcs like arc DE?



Ex. 83. The adjoining figure represents a *segmental* arch. The method of construction and the dimensions are indicated in the figure. If the arch is made of 11 stones of equal size, what is the length of the arc XY? What is the height of the arch?

Ex. 84. What is the total pressure on a locomotive engine piston which is 20 in. in diameter, if the pressure of the steam is 120 lb. per square inch?

Ex. 85. The diameter of the larger of two pulley wheels is 20 in. and of the smaller is 8 in. If the smaller wheel revolves 500 times per minute, how many times per minute does the larger revolve?

Ex. 86. What is the area of the segment of the circle whose area is 144 π sq. in., if the angle which intercepts the arc of the segment is 60°?

Note. Additional Exercises 46 to 73, p. 307, can be studied now.

Miscellaneous Review

Ex. 87. How large is each vertex angle of a regular octagon?

Ex. 88. Why has a regular polygon a center, and what is it?

Ex. 89. Why has a regular polygon an apothem and what is it?

Ex. 90. How long is the apothem of a regular hexagon inscribed in a circle of radius 14 in.?

Ex. 91. How large is the central angle of a regular 12-gon?

Ex. 92. How large is the central angle of a regular n-gon?

Ex. 93. If the mid-point of each arc subtended by the sides of a regular inscribed hexagon is joined to each end of its arc, what kind of polygon is formed by the resulting chords?

Ex. 94. If tangents be drawn to a circle at the vertices of a regular inscribed octagon, what kind of polygon is formed by them?

Ex. 95. What is the area of the regular pentagon whose side is 10 in. and whose apothem is 15.4 in.?

Ex. 96. In a circle of radius 2 in., inscribe a regular octagon.

Ex. 97. Compute the apothem of the square inscribed in a circle of radius 2 in.

Ex. 98. In a circle of radius 1.5 in. inscribe a regular 12-gon.

Ex. 99. Compute the apothem of the regular triangle inscribed in a circle of radius 1.5 in.

Ex. 100. The perimeter of one of two regular pentagons is 80 in. What is the perimeter of the second if a side of the second is $\frac{3}{2}$ of a side of the first?

Ex. 101. The area of one of two regular 15-gons is 384 sq. in. What is the area of the second if a side of the second is $\frac{1}{2}$ the side of the first?

Ex. 102. Define: (*a*) variable; (*b*) constant; (*c*) limit of a variable.

Ex. 103. If $x \to 20$, what is the limit of $\frac{3}{5}x$?

Ex. 104. Variables x and y are equal, and $x \to 18$. What is the limit of y?

Ex. 105. Define: (*a*) length of a circle; (*b*) area of a circle.

Ex. 106. In a circle of radius 9 in., regular polygons of 6, 12, 24, 48, etc. sides are inscribed. What is the limit of the apothems of these polygons?

Ex. 107. If the length and area of a circle are 18π in. and 81π sq. in. respectively; if, in it there are inscribed regular polygons of 4, 8, 16, 32, etc. sides; what is the limit of the perimeters and of the areas of these polygons?

Ex. 108. The radii of two circles are 3 in. and 12 in., respectively. The second circle is how many times as long as the first?

Ex. 109. In Ex. 108, how does the area of the first circle compare with the area of the second?

Ex. 110. What is the definition of π? Give two approximations to its value.

Ex. 111. How long is the circle of radius 3 in.? Of diameter 16 in.?

Ex. 112. What is the area of each of the circles in Ex. 111?

Ex. 113. What is the area of the sector whose angle is 216° in a circle whose area is 25π sq. in.?

OPTIONAL TOPICS

Four optional topics follow. As has been described on pages 86, 134, 184, and 220, these topics are not explicitly required by the College Entrance Examination Board, and, for that reason, may very properly be omitted from a minimum course, without implying in the least any disbelief in the educational value or the mathematical interest of the topics.

Each topic is independent of the others. Teachers should feel free to select the group or groups which appear to meet the needs of the class.

Topic A. **Construction of Other Regular Polygons.**

Current recommendations exclude from the minimum course the construction of the decagon, pentagon, and related regular polygons. Many teachers of experience will want at least their good and excellent pupils to have the opportunity to study this subject matter. In fact, the construction of the pentagon, and the special study of it, might reasonably be called very practical geometry.

Topic B. **Inscription of Circles.**

A topic of considerable practical value because of its application in artistic design.

Topic C. **Determination of the Value of π.**

Topic D. **Trigonometric Solution of Problems about Regular Polygons.**

Optional Topic A

CONSTRUCTION OF OTHER REGULAR POLYGONS

385. Before considering what other regular polygons can be constructed by ruler and compasses alone, we must study what has been called Golden Section of a segment.

A segment is divided internally by a given point in **extreme and mean ratio** when the whole segment is to the longer part as the longer part is to the shorter.

Thus C divides AB internally in extreme and mean ratio if

$$AB : AC = AC : CB.$$

PROPOSITION XI. PROBLEM

386. *Divide a given segment in extreme and mean ratio.*

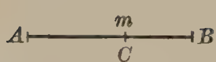


Fig. 1

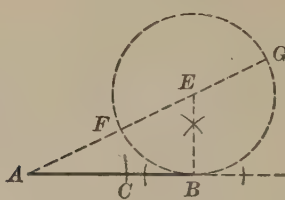


Fig. 2

Given segment $AB = m$.

Required to divide AB in extreme and mean ratio.

Analysis. 1. Let $x = AC$ and $\therefore m - x = CB$.

2. $\therefore m : x = x : (m - x)$. § 385

3. $\therefore x^2 = m(m - x)$.

4. $\therefore x^2 + mx = m^2$.

5. $\therefore x^2 + mx + (\frac{1}{2}m)^2 = m^2 + (\frac{1}{2}m)^2$.

6. $\therefore (x + \frac{1}{2}m)^2 = m^2 + (\frac{1}{2}m)$.

7. $\therefore x + \frac{1}{2}m$ is the hypotenuse of a right triangle whose base is m and whose altitude is $\frac{1}{2}m$.

Construction. (Fig. 2.) 1. Draw $EB \perp AB$, making $AB = m$ and $EB = \frac{1}{2}m$.

2. Draw AE and on it take $EF = EB = \frac{1}{2}m$.

3. On AB, take $AC = AF$.

Statement. $AB : AC = AC : CB$.

Proof:

STATEMENTS	REASONS
1. Draw the ⊙ with center E and radius EB. Extend AE, cutting the ⊙ at G.	1. Possible constructions.
2. AB is tangent to ⊙ BFG.	2. Why?
3. $\therefore \dfrac{AG}{AB} = \dfrac{AB}{AF}$.	3. § 287.
4. $\therefore \dfrac{AG}{AB} = \dfrac{AB}{AC}$.	4. $AC = AF$.
5. $\therefore \dfrac{AG - AB}{AB} = \dfrac{AB - AC}{AC}$.	5. § 258.
6. But $AB = 2\,EB = FG$.	6. Construction.
7. $\therefore AG - AB = AG - FG = AF$, or AC.	7. Ax. 2, § 49.
8. $\therefore \dfrac{AC}{AB} = \dfrac{CB}{AC}$.	8. Ax. 2, § 49.
9. $\therefore \dfrac{AB}{AC} = \dfrac{AC}{CB}$.	9. § 256.

Note. "Golden Section" represented to the Greeks the most artistic division of a segment. A rectangle having base AB and altitude AC is a good-appearing rectangle; also one having base AC and altitude BC. These dimensions correspond closely to dimensions given by a rule often employed now in art work; namely, for the altitude take a length which is a little more than half and a little less than two thirds of the base. If the equation of Step 6 of the Analysis is solved for x:

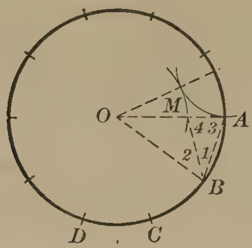
$$(x + \tfrac{1}{2}m)^2 = \tfrac{5}{4}m^2.$$

$$x = \pm\sqrt{\frac{5\,m^2}{4}} - \frac{m}{2} \text{ or } x = \pm\frac{m}{2}\sqrt{5} - \frac{m}{2}.$$

$$\therefore x = \frac{m}{2}(\sqrt{5} - 1) = \frac{m}{2}(2.236 - 1) = \frac{m}{2}(1.236) = .6\,m.$$

PROPOSITION XII. PROBLEM

387. *Inscribe a regular decagon in a given circle.*



Given $\odot\ ACD$.

Required to inscribed a regular decagon in $\odot ACD$.

Construction. 1. Draw radius OA. Divide it at M in extreme and mean ratio so that $OA : OM = OM : AM$.

2. With A as center and OM (the longer segment) as radius, draw an arc cutting the given circle at B.

Statement. AB is the side of the required decagon.

Proof:

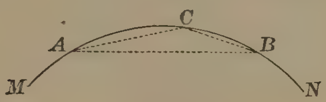
STATEMENTS	REASONS
1. Draw OB and BM.	1. Possible.
In $\triangle OAB$ and $\triangle ABM$:	
2. $\angle A = \angle A$;	2. An $\angle =$ itself.
3. Since $OA : OM = OM : AM$.	3. Construction.
4. $\therefore OA : AB = AB : AM$.	4. $AB = OM$. Ax. 2.
5. $\therefore \triangle OAB \sim \triangle ABM$.	5. § 280.
6. $\therefore \triangle ABM$ is isosceles.	6. The $\triangle$s are $\sim$.
7. $\therefore BM = AB = OM$.	7. Ax. 2.
8. $\angle 4 = 2 \cdot \angle AOB$.	8. Why?
9. $\therefore \angle 3 = 2 \cdot \angle AOB$.	9. Since $\angle 3 = \angle 4$.
10. $\therefore \angle ABO = 2 \cdot \angle AOB$.	10. Since $\angle ABO = \angle 3$.
11. $\angle AOB + \angle 3 + \angle ABO = 180°$.	11. Why?
12. $\therefore 5 \cdot \angle AOB = 180°$.	12. Ax. 2.
13. $\therefore \angle AOB = 36°$.	13. § 209.
14. $\therefore AB$ is one side of the regular inscribed decagon.	14. Why?

388. Cor. 1. *Chords joining the alternate vertices of a regular inscribed decagon, starting with any vertex, form a regular inscribed pentagon.*

389. Cor. 2. *Regular inscribed polygons of 20, 40, 80, etc., sides can be constructed with ruler and compasses alone.*

PROPOSITION XIII. PROBLEM

390. *Inscribe a regular pentadecagon* (15-*gon*) *in a circle.*



Given $\odot MN$.

Required to inscribe in $\odot MN$ a pentadecagon.

Analysis.

1. The central $\angle$ of a pentadecagon $= \dfrac{360°}{15} = 24°$.

2. But $24° = 60° - 36°$.

3. $\therefore$ combine the constructions of § 363 and § 387.

Construction. Construct AB, a side of a regular inscribed hexagon, and AC, a side of a regular inscribed decagon. Draw chord BC.

Statement. BC is a side of the regular inscribed 15-gon.

Proof. $\widehat{BC} = (\frac{1}{6} - \frac{1}{10})$ or $\frac{1}{15}$ of the circle. Const.

391. Cor. *Regular polygons of 30, 60, etc., sides can be inscribed in a circle by ruler and compasses alone.*

392. Summary. Combining the results of §§ 361–365, and 387–391, a regular polygon can be inscribed by use of ruler and compasses alone if the number of its sides is one of the following numbers: (*a*) 4, 8, 16, 32, etc.; (*b*) 3, 6, 12, 24, etc.; (*c*) 5, 10, 20, etc.; (*d*) 15, 30, 60, etc.

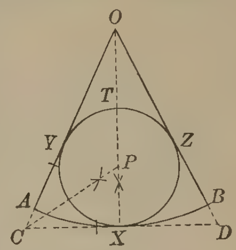
Observe that there are many numbers which are missing. The polygon of 17 sides also can be constructed. This was proved by the great mathematician Gauss.

Optional Topic B

INSCRIPTION OF CIRCLES

393. Inscription of circles within regular polygons and within circles is a characteristic feature of art window and other designs. Many of the necessary constructions are based upon the following illustrative problem, or may be discovered by means of an analysis similar to that employed in this problem.

Problem. Inscribe a circle in a given sector of a circle.



Analysis. 1. Let ⊙ XYZ be tangent to radius OB at Z, to OA at Y, and to arc AXB at X. Let CD be the common tangent to arc AXB and ⊙ XYZ at X.

2. Then ⊙ XYZ is inscribed in $\triangle OCD$.

3. Hence the center P of ⊙ XYZ lies on the bisectors of $\angle COD$ and $\angle OCD$.

4. The radius is the distance from P to X.

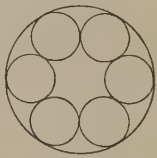
5. The construction is evident at once.

Ex. 114. Inscribe a circle in a sector whose angle is 60°, in a circle of radius 2 in.

Ex. 115. Construct a circle with radius 2 in.; and within it construct a sector whose angle is 90°.

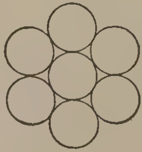
(*a*) Within this sector inscribe a circle.

(*b*) Compare the area of this circle with the area of the sector itself when the radius of the given circle is r instead of 2.



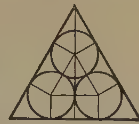
Ex. 116. Construct six equal circles within a circle of radius 2 in., each tangent internally to the given circle and tangent externally to two of the inner circles.

Ex. 117. Construct a circle which will be tangent to each of the circles constructed in Ex. 116.



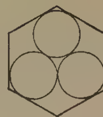
Ex. 118. Construct six equal circles tangent externally to a circle of radius $\frac{1}{2}$ in. such that each circle is also tangent externally to two of the constructed circles.

Ex. 119. Construct a circle which will be tangent to each of the six circles constructed in Ex. 118.



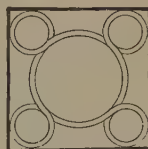
Ex. 120. In an equilateral triangle inscribe three equal circles, each tangent to two sides of the triangle and tangent externally to the other two circles.

Ex. 121. In a regular hexagon inscribe six equal circles, each tangent to two sides of the hexagon and tangent externally to two of the circles.

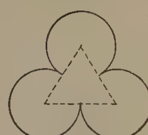


Ex. 122. Inscribe in a regular hexagon three equal circles, each tangent to two sides of the hexagon and tangent externally to two circles.

Ex. 123. In a regular octagon inscribe four equal circles, each tangent to two sides of the octagon and also tangent externally to two circles.



Ex. 124. The adjoining design appears in a floor pattern in a corridor of the Congressional Library. Construct such a figure, making a 5-in. square, the radius of the smallest ⊙ 1.5 in., and the radius of the concentric ⊙ 1.75 in.



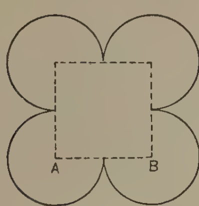
Ex. 125. The adjoining curve is a *trefoil.*

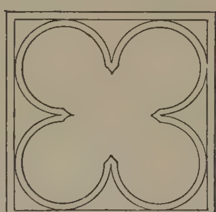
(*a*) Construct such a figure based upon an equilateral triangle whose side is 2 in. long.

(*b*) What is the length of the trefoil if the side of the equilateral triangle is s inches?

(*c*) What is the area within the trefoil if the length of the side of the equilateral triangle is s inches?

Ex. 126. The curved-line figure below at the left is a *quatrefoil.*





(*a*) Construct a quatrefoil based upon a square whose side is 2 in.

(*b*) What is the length of the quatrefoil if $AB = s$ inches?

(*c*) What is the area within the quatrefoil if $AB = s$ inches?

(*d*) Notice that the quatrefoil is used in the adjoining design.

Note. Additional Exercises 74 to 77, page 310, can be studied now.

Optional Topic C

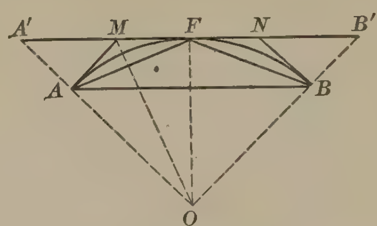
COMPUTATION OF THE CONSTANT π

394. An approximate value of π can be found by computing the perimeter of a regular polygon of very many sides in a circle of radius 1. This computation is rendered easy by the following proposition.

Note. Obviously the following three pages should merely be read over for the purpose of securing comprehension of the procedure, — not in the expectation that anyone will be required to reproduce the demonstrations or the computations.

PROPOSITION XIV. PROBLEM

395. Given p_n and P_n, the perimeters of the regular inscribed and of the regular circumscribed n-gons; find p_{2n} and P_{2n}, the perimeters of the regular inscribed and the regular circumscribed $2n$-gons.



(*a*) To find P_{2n}.

Solution. 1. Let AB be one side of the regular inscribed n-gon and F the mid-point of arc AB.

2. Draw the tangent to the circle at F, meeting OA and OB extended at A' and B' respectively.

Then $A'B'$ is one side of the regular circumscribed n-gon. Also $A'F = \frac{1}{2} A'B'$, and hence $P_n = 2\,nA'F$.

3. Let tangents to the circle at A and B meet $A'B'$ at M and N respectively. Then MN is one side of the regular circumscribed $2n$-gon; $MF = \frac{1}{2}MN$ and $P_{2n} = 4\,nMF$.

(Continued on page 263)

I. To find P_{2n}.

1. $A'M : MF = OA' : OF$.	1. § 271.
2. But $P_n : p_n = OA' : OF$.	2. § 367.
3. $\therefore P_n : p_n = A'M : MF$.	3. Why?
4. $\therefore (P_n + p_n) : p_n = (A'M + MF) : MF$.	4. § 257.
5. $\therefore (P_n + p_n) : p_n = A'F : MF$.	5. Why?
6. $\therefore (P_n + p_n) : p_n = 4\,nA'F : 4\,nMF$.	6. Why?
7. $\therefore (P_n + p_n) : p_n = 2\,P_n : P_{2n}$.	7. $P_n = 2\,nA'F$.
8. $\therefore P_{2n}(P_n + p_n) = 2\,P_n \cdot p_n$.	8. Why?
9. $\therefore P_{2n} = \dfrac{2\,P_n \cdot p_n}{P_n + p_n}$.	9. Why?

II. To find p_{2n}.

1. $\triangle ABF$ and $\triangle AFM$ are isos. ⧍.	1. Why?
2. $\therefore \triangle ABF \sim \triangle AFM$.	2. Give the proof.
3. $\therefore AB : AF = AF : MF$.	3. Why?
4. $\therefore \overline{AF}^2 = AB \times MF$.	4. Why?
5. But $AF = \dfrac{p_{2n}}{2\,n}$; $AB = \dfrac{p_n}{n}$; $MF = \dfrac{P_{2n}}{4\,n}$.	5. See solution (2) and (3).
6. $\therefore \left(\dfrac{p_{2n}}{2\,n}\right)^2 = \left(\dfrac{p_n}{n} \times \dfrac{P_{2n}}{4\,n}\right)$,	6. Ax. 2.
or $\dfrac{p_{2n}^2}{4\,n^2} = \dfrac{p_n \times P_{2n}}{4\,n^2}$.	7. Why?
7. $\therefore p_{2n}^2 = p_n \times P_{2n}$.	
8. $\therefore p_{2n} = \sqrt{p_n \times P_{2n}}$.	8. Why?

Note. The formulas $P_{2n} = \dfrac{2\,P_n p_n}{P_n + p_n}$ and $p_{2n} = \sqrt{p_n P_{2n}}$ are quite remarkable. If the perimeters of the regular inscribed and regular circumscribed polygons of say 4 sides are known, then the perimeters of the regular circumscribed and regular inscribed polygons of 8 sides can be computed by mere substitution; then those of 16 sides; and so on. Therefore the perimeters of inscribed and circumscribed regular polygons of very many sides can be computed.

PROPOSITION XV. PROBLEM

396. *Compute an approximate value of π.*

Solution. 1. If the diameter of a circle is 1, the side of an inscribed square is $\frac{1}{2}\sqrt{2}$, and hence the perimeter of the square is $2\sqrt{2}$, or $p_4 = 2.82843$.

2. The side of a circumscribed square is 1, and $P_4 = 4$.

3. $$P_8 = \frac{2\,P_4 \times p_4}{P_4 + p_4}. \qquad \text{§ 395, I.}$$

Hence $$P_8 = \frac{2 \times 4 \times 2.82843}{4 + 2.82843} = 3.31371.$$

4. $$p_8 = \sqrt{p_4 \times P_8}. \qquad \text{§ 395, II.}$$

Hence $$p_8 = \sqrt{2.82843 \times 3.31371} = 3.06147.$$

5. Similarly $$P_{16} = \frac{2\,P_8 \times p_8}{P_8 + p_8} = \frac{2 \times 3.31371 \times 3.06147}{3.31371 + 3.06147}$$

or $P_{16} = 3.18260$.

And $p_{16} = \sqrt{p_8 \times P_{16}} = \sqrt{3.06147 \times 3.18260} = 3.12145$.

6. In this manner, the following table may be computed:

No. of Sides	Perimeter of Reg. Circ. Polygon	Perimeter of Reg. Insc. Polygon
4	4.	2.82843
8	3.31371	3.06147
16	3.18260	3.12145
32	3.15172	3.13655
64	3.14412	3.14033
128	3.14222	3.14128
256	3.14175	3.14151
512	3.14163	3.14157

7. The last results show that the circumference of the circle whose diameter is 1 > 3.14157 and < 3.14163.

Note. Read the note following § 394.

8. But $C = \pi d$, and when $d = 1$, $C = \pi$.

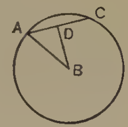
Hence an approximate value of π is 3.1416, correct to the fourth decimal place.

Optional Topic D

TRIGONOMETRIC SOLUTION OF PROBLEMS

Ex. 127. If AC is a side of a regular decagon inscribed in a circle of radius 12 in., how long is AC and the apothem BD?

Solution. 1. Since $AC = \frac{1}{10} \times 360°$, $\angle DBA = 18°$.



2. $\therefore \dfrac{BD}{AB} = \cos \angle DBA$; or $BD = 12 \times \cos 18°$.

3. $\therefore BD = 12 \times .951$, or 11.4^+.

4. $\dfrac{AD}{AB} = \sin \angle DBA$; or $AD = 12 \times \sin 18°$.

5. $\therefore AD = 12 \times .309$; or $AD = 3.7^+$.

6. Since $AD = 3.7$, $AC = 7.4$.

Ex. 128. Find the apothem and a side of the regular pentagon inscribed in a circle of radius 10 in.

Ex. 129. Find the side and apothem of the regular octagon inscribed in a circle of radius 6 in.

Find the perimeter and the area of the following regular polygons:

Ex. 130. The regular 12-gon inscribed in a circle of radius 8 in.

Ex. 131. The regular decagon inscribed in a circle of radius 5 in.

Ex. 132. The regular octagon circumscribed about a circle of radius 8 in.

Ex. 133. The regular pentagon circumscribed about the circle of radius 9 in.

Ex. 134. The regular decagon circumscribed about the circle of radius 10 in.

Ex. 135. The regular 12-gon circumscribed about the circle of radius 7 in.

Ex. 136. Find the side of the regular hexagon whose apothem is 4.

Ex. 137. Find the side of the regular octagon whose apothem is 5.

Ex. 138. Find the side of the regular decagon whose apothem is 6.

Ex. 139. Find the side of the equilateral triangle whose altitude is 8.

Ex. 140. Find the altitude of the equilateral triangle whose side is 9.

Ex. 141. Find the radius of the regular pentagon whose side is 5.

Ex. 142. Find the radius of the regular octagon whose side is 4.

Ex. 143. Find the radius of the regular hexagon whose apothem is 3.

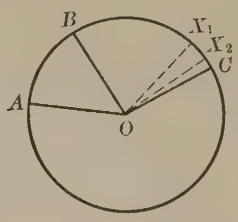
Ex. 144. Find the radius of the regular decagon whose side is 2.5.

APPENDIX

A. The Incommensurable Cases

397. There are two incommensurable cases. These propositions complete the proofs of the theorems given in § 207 and § 262. If it is desired to read § 397 when studying § 207, then it will be necessary to read also § 370 and § 371, which give an introduction to the theory of limits.

398. *In the same circle or in equal circles, central angles have the same ratio as their intercepted arcs.* (When the angles are incommensurable.)



Hypothesis. In $\odot ABC$, $\angle AOB$ and $\angle BOC$ are two incommensurable central angles intercepting the arcs AB and BC respectively.

Conclusion. $$\frac{\angle BOC}{\angle AOB} = \frac{\widehat{BC}}{\widehat{AB}}.$$

Note. Pages 266–271 contain material which is seldom studied by classes. However, there may be individual pupils in high schools or students in classes in higher geometry who should have an opportunity to study this subject matter. For such pupils is it included in this text.

Proof. 1. Divide $\angle AOB$ into two equal parts and let one of these be applied as unit of measure to $\angle BOC$.

2. Since $\angle AOB$ and $\angle BOC$ are incommensurable, a certain number of angles equal to $\frac{1}{2}\angle AOB$ will equal $\angle BOX_1$, leaving a remainder $\angle X_1OC$ which is less than the unit of measure.

3. $\angle AOB$ and $\angle BOX_1$ are commensurable.

4. $$\frac{\angle BOX_1}{\angle AOB} = \frac{\overset{\frown}{BX_1}}{\overset{\frown}{AB}}. \qquad \S\, 207$$

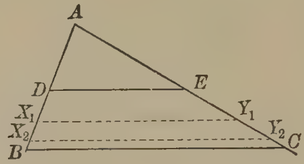
5. Take now as unit of measure $\frac{1}{4}\angle AOB$. This measure will be contained an integral number of times in $\angle AOB$ and also in $\angle BOX_1$; further, the unit of measure *may* be contained once in $\angle X_1OC$, leaving a remainder $\angle X_2OC$ which is less than the new unit of measure.

Again $$\frac{\angle BOX_2}{\angle AOB} = \frac{\overset{\frown}{BX_2}}{\overset{\frown}{AB}}. \qquad \S\, 207$$

6. Continue in this manner to decrease indefinitely the unit of measure. The remainder $\angle XOC$, being always less than the unit of measure, will approach the limit O.

7. $\therefore \angle BOX \to \angle BOC.$	7. § 370, (*c*).
8. $\therefore \dfrac{\angle BOX}{\angle AOB} \to \dfrac{\angle BOC}{\angle AOB}.$	8. § 371, (*a*).
9. $\overset{\frown}{BX} \to \overset{\frown}{BC}.$	9. § 370, (*c*).
10. $\therefore \dfrac{\overset{\frown}{BX}}{\overset{\frown}{AB}} \to \dfrac{\overset{\frown}{BC}}{\overset{\frown}{AB}}.$	10. § 371, (*a*).
11. But $\dfrac{\angle BOX}{\angle AOB} = \dfrac{\overset{\frown}{BX}}{\overset{\frown}{AB}}.$	11. Steps 4, 5.
12. $\therefore \dfrac{\angle BOC}{\angle AOB} = \dfrac{\overset{\frown}{BC}}{\overset{\frown}{AB}}.$	12. § 371, (*b*).

399. *A parallel to one side of a triangle divides the other two sides proportionally, when the segments of one side are incommensurable.*



Hypothesis. In $\triangle ABC$, segments AD and BD are incommensurable; $DE \parallel BC$, meeting AC at E.

Conclusion. $BD : AD = CE : AE$.

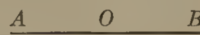
Proof:

Statements	Reasons
1. Divide AD into any number of equal parts (say two), and apply one of these parts to BD as unit of measure.	
2. Since AD and BD are incommensurable, a certain number of segments equal to the unit of measure will extend from D to X_1, leaving a remainder X_1B which is less than the unit of measure.	
3. Draw $X_1Y_1 \parallel BC$, meeting AC at Y_1. Then $DX_1 : AD = EY_1 : AE$.	§ 262
4. Take now as unit of measure $\frac{1}{4}$ AD. This measure will be contained an integral number of times in AD and also in DX_1; further, the unit of measure may be contained once in X_1B, leaving a remainder X_2B which is less than the new unit of measure. Draw $X_2Y_2 \parallel BC$, meeting AC at Y_2. Then $DX_2 : AD = EY_2 : AE$.	§ 262
5. Continue to decrease the unit of measure. Then $XB \to 0$, since $XB <$ the unit of measure.	
6. $\therefore DX \to DB$, and $DX : AD \to DB : AD$.	§ 371, (*a*)
7. $EY \to EC$ and $EY : AE \to EC : AE$.	§ 371, (*a*)
8. $\therefore DB : AD = EC : AE$.	§ 371, (*b*)

B. Symmetry in Plane Figures

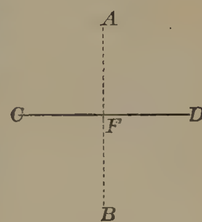
400. Two points are symmetrical with respect to a third point, called the center of symmetry, when the latter bisects the segment which joins them.

Thus, if O is the mid-point of segment AB, points A and B are symmetrical with respect to O as center.

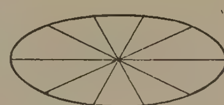


401. Two points are symmetrical with respect to a straight line, called the axis of symmetry, when the latter bisects at right angles the segment which joins them.

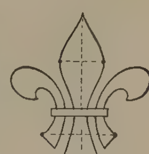
Thus, if CD bisects segment AB at right angles, points A and B are symmetrical with respect to CD as an axis.



402. A figure is symmetrical with respect to a center when every straight line drawn through the center cuts the figure in two points which are symmetrical with respect to that center.



403. A figure is symmetrical with respect to an axis when every straight line perpendicular to the axis cuts the figure in two points which are symmetrical with respect to that axis.



Ex. 145. Does a circle have a center of symmetry?

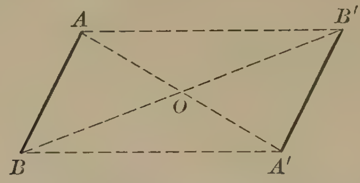
Ex. 146. (*a*) Locate upon a sheet of paper a point O and four other points X, Y, Z, and W.

(*b*) Construct the points X', Y', Z', and W', which are symmetrical respectively to X, Y, Z, and W, with respect to O as center.

Ex. 147. (*a*) Draw any straight line AB of indefinite length and upon one side of it locate at random points X, Y, and Z.

(*b*) Construct the points X', Y', and Z', which are respectively symmetrical to X, Y, and Z, with respect to AB as axis.

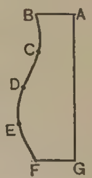
404. Theorem. *Two segments which are symmetrical with respect to a center are equal and parallel.*



Hypothesis. Segments AB and $A'B'$ are symmetrical with respect to center O.

Conclusion. AB and $A'B'$ are equal and parallel.

Proof: STATEMENTS	REASONS
1. Draw lines AA' and BB' intersecting at O; draw AB' and $A'B$.	§ 400
2. O bisects AA' and BB'.	Prove it.
3. $\therefore AB'A'B$ is a $\square$.	Prove it.
4. $\therefore AB$ and $A'B'$ are equal and parallel.	



Ex. 148. (*a*) Draw a figure something like the adjoining one.

(Let AB and FG be perpendicular to AG, and let $BCDEF$ be a curved line.)

(*b*) Construct the figure symmetrical to $ABCDEFG$ with respect to AG as axis.

Ex. 149. Prove that two segments which are equal and parallel are symmetrical with respect to a center.

Ex. 150. Prove that the bisector of the vertical angle of an isosceles triangle is an axis of symmetry of the triangle.

Ex. 151. How many axes of symmetry does an equilateral triangle have?

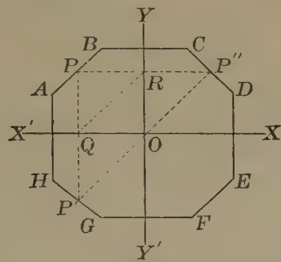
Ex. 152. Prove that the intersection of the diagonals of a parallelogram is the center of symmetry of the parallelogram.

Ex. 153. Does a rhombus have a center of symmetry?

Ex. 154. Does the rhombus have an axis of symmetry?

Ex. 155. Does a rectangle have an axis of symmetry?
Does it have a second axis of symmetry?
Does it have a center of symmetry?

405. Theorem. *If a figure is symmetrical with respect to each of two perpendicular axes, it is symmetrical with respect to their intersection as center.*



Hypothesis. Figure AE is symmetrical with respect to axes XX' and YY'. XX' is perpendicular to YY' at O.

Conclusion. AE is symmetrical with respect to O as center.

Proof:

STATEMENTS	REASONS
1. From P' any point of AE draw $P'Q \perp$ to XX' at Q, and meeting AE again at P. Then $PQ = P'Q$.	1. Why?
2. From P draw $PR \perp YY'$ meeting YY' at R and meeting AE again at P''. Then $PR = RP''$.	2. Why?
3. Draw $P'P''$.	3. Possible.
4. $YY' \parallel PP'$ and bisects PP''. $\therefore YY'$ passes through the mid-point of $P'P''$.	4. Why?
5. Similarly, XX' passes through the mid-point of $P'P''$.	5. Proof like Steps 1–4.
6. Hence O, the intersection of XX' and YY', must be the mid-point of $P'P''$.	6. Post. 4.
7. $\therefore$ AE is symmetrical with respect to O as center.	7. § 402.

Ex. 156. Answer for a square the questions proposed in Ex. 155.

Ex. 157. Answer for an isosceles trapezoid the same questions.

Ex. 158. Answer for a regular hexagon the same questions.

Ex. 159. Answer for a regular pentagon the same questions.

ADDITIONAL EXERCISES

BOOK I

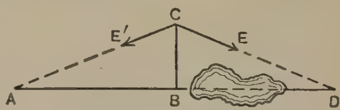
Ex. 1. Two quadrilaterals are congruent if three sides and the two included angles of the one are equal respectively to three sides and the two included angles of the other.

Suggestion. Prove this theorem by superposition.

Ex. 2. Two quadrilaterals are congruent if three angles and the two included sides of the one are equal respectively to three angles and the two included sides of the other.

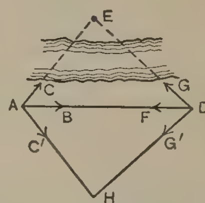
Ex. 3. Prove that the bisectors of two corresponding angles of two congruent triangles are equal.

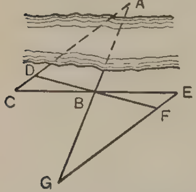
Ex. 4. Explain why the inaccessible distance BD of the adjoining figure can be found by measuring AB, if AB is obtained as follows:



At B, erect a vertical rod BC, to the top of which is hinged a rod CE; point CE at D, and observe $\angle BCE$; revolve BCE around BC, keeping BC vertical, until CE' points at a position A in DB extended, and so that $\angle BCE' = \angle BCE$.

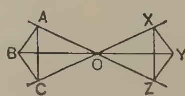
Ex. 5. Explain how the hinged rods used in Ex. 4 can be used to produce $\triangle ADH$ which shall be congruent to $\triangle ADE$, and thus determine AE by measuring AH.

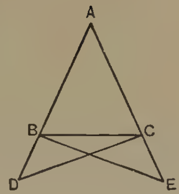




Ex. 6. Explain why the following construction makes $BG = AB$. Through B, lay off CBE so that $CB = BE$. Between C and A, take D. On DB extended, take F so that $DB = BF$. Extend EF until it cuts AB extended at G. Prove $AB = BG$.

Ex. 7. In the adjoining figure, if AO, BO, and CO are extended to Z, Y, and X respectively, so that $AO = OZ$, $BO = OY$, and $CO = OX$, then $\triangle ABC \cong \triangle ZYX$.



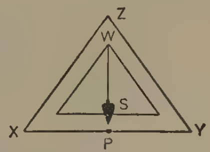


Ex. 8. Prove that the base angles of an isosceles triangle are equal, using the following construction.

Hypothesis. $AB = AC$.

Conclusion. $\angle ABC = \angle ACB$.

Construction. Extend AB to D. Extend AC to E, making $CE = BD$. Draw DC and BE.

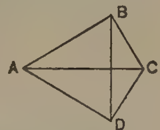


Ex. 9. In the adjoining figure, $XZ = ZY$ and $PX = PY$. From point W, a plumb bob hangs. Explain why XY is level when the plumb bob S points directly at P.

Ex. 10. Construct the angle which is double $\angle B$ of Ex. 64, Book I.

Ex. 11. Construct the sum of $\angle A$ and $\angle B$ of Ex. 64, Book I.

Ex. 12. Construct an isosceles triangle having its equal sides 3 in. in length and the angle included by them equal to $\angle B$ given in Ex. 64.



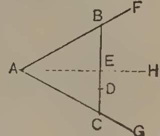
Ex. 13. If a diagonal of a quadrilateral $ABCD$ bisects two of its angles, it is perpendicular to the other diagonal and bisects it.

Ex. 14. Construct the perpendicular-bisector of a segment taken along the lower edge of the paper.

Ex. 15. Prove that the medians to two corresponding sides of congruent triangles are equal.

Ex. 16. If two triangles have two sides and the median to one of them equal respectively to two sides and the corresponding median of the other, the triangles are congruent.

Ex. 17. Draw any angle and construct its bisector. Through its vertex, construct a line perpendicular to the bisector. Prove that this last line makes equal angles with the sides of the given angle.

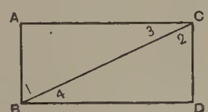


Ex. 18. Construct a line through a given point within a given acute angle, which will form with the sides of the given angle an isosceles triangle.

Suggestion. How must CB intersect AH?

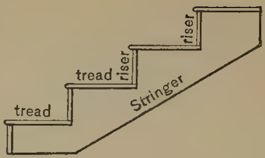
Ex. 19. Prove Cor. 1 (§ 95) if $\angle 4 = \angle 8$.

Ex. 20. Prove Cor. 3 (§ 97) if $\angle 3 + \angle 5 = 1$ st. $\angle$.

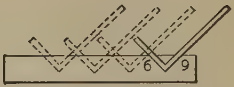


Ex. 21. Prove that $AB \parallel CD$ (Fig. § 95) if $\angle 4 + \angle 7 = 1$st. $\angle$.

Ex. 22. If $AB = CD$ and $\angle 1 = \angle 2$ in the adjoining figure, prove $AB \parallel CD$ and also $AC \parallel BD$.



Ex. 23. In building stairs two or more *stringers* are required. To make a stringer having a 9-in. *tread* and a 6-in. *riser*, a carpenter uses his square as in the figure below. For each step he places his square so that the 9-in. mark and the 6-in. mark fall along the edge of the board from which he is cutting the stringer.



Will the treads all be parallel? Why?

Will the risers all be parallel? Why?

Will each riser be perpendicular to its tread? Why?

Ex. 24. One triangle used by draughtsmen has an angle of 90° and an angle of 60°. Why should it be called a "60–30" triangle?

Ex. 25. The other triangle used by draughtsmen has an angle of 90° and an angle of 45°. How large is the remaining angle?

Ex. 26. In a $\triangle ABC$, if $\angle A = 90°$, and $\angle B = \angle C$, how large are $\angle B$ and $\angle C$?

Ex. 27. Find the three angles of a triangle if the second is four times the first, and the third is seven times the first. (Algebraic solution.)

Ex. 28. Find the three angles of a triangle if the second exceeds the first by 40°, and the third exceeds the second by 40°.

Ex. 29. The vertical angle of an isosceles triangle is n degrees. Express each of the base angles.

Ex. 30. One base angle of an isosceles triangle is n degrees. Express each of the other angles of the triangle.

Ex. 31. Determine by construction the angle C of a $\triangle ABC$ if $\angle A$ and $\angle B$ are the angles given in Ex. 64, Book I.

Ex. 32. Prove that two isosceles triangles are congruent when the vertical angle and the base of one are equal respectively to the vertical angle and the base of the other.

Suggestion. Prove the corresponding base angles also are equal.

Ex. 33. If one acute angle of a right triangle is 35°, how large is the other acute angle?

Ex. 34. If perpendiculars be drawn from any point in the base of an isosceles triangle to the equal sides, they make equal angles with the base.

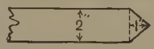
Ex. 35. Prove that the altitude drawn to the hypotenuse of a right triangle divides the right angle into two parts which are equal respectively to the acute angles of the right triangle.

Ex. 36. If two opposite angles of a quadrilateral are equal and if the diagonal joining the other two angles bisects one of them, then it bisects the other also.

Ex. 37. Prove that the altitudes drawn to two corresponding sides of two congruent triangles are equal.

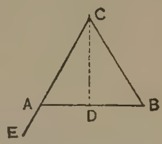
Ex. 38. In each of two sides of a triangle, find the point which is equidistant from the ends of the third side.

Ex. 39. Construct a pattern for the pointed end of a belt, assuming that the belt material is 2 in. wide, and that the point is to project 1 in. beyond the square end of the belt.



Ex. 40. Draw any straight line of indefinite length and select two points not in it. Find the point in the line which is equidistant from the two given points.

Ex. 41. Find a point in one side of a triangle which is equidistant from the other two sides of the triangle.



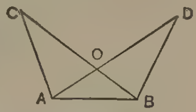
Ex. 42. Prove that either exterior angle at the base of an isosceles triangle is equal to the sum of a right angle and one half the vertical angle.

Ex. 43. If from the vertex of one of the equal angles of an isosceles triangle a perpendicular be drawn to the opposite side, it makes with the base an angle equal to one half the vertical angle of the triangle.

Ex. 44. $\triangle ABC$ is an equilateral triangle. BP, the bisector of $\angle B$, meets AC at P; CM, the bisector of exterior angle ACR, meets BP extended at M. MN is perpendicular to CR. Prove $MN = BP$.

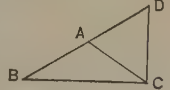
Ex. 45. If the equal sides of an isosceles triangle be extended beyond the base, the bisectors of the exterior angles so formed form with the base another isosceles triangle.

Ex. 46. If $\triangle ABC$ and $\triangle ABD$ are two triangles on the same base and on the same side of it, such that $AC = BD$ and $AD = BC$, and AD and BC intersect at O, then $\triangle OAB$ is isosceles.



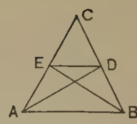
Ex. 47. If equiangular triangles be constructed upon the sides of any triangle, the lines drawn from their outer vertices to the opposite vertices of the given triangle are equal.

Ex. 48. If the angle at the vertex of isosceles $\triangle ABC$ is equal to twice the sum of the equal angles B and C, and if CD is perpendicular to BC, meeting BA extended at D, prove $\triangle ACD$ is equilateral.



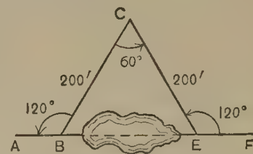
Suggestion. Let x = the number of degrees in $\angle B$.

Ex. 49. If the bisectors of the equal angles of an isosceles triangle meet the equal sides at D and E respectively, prove that DE is parallel to the base of the triangle.

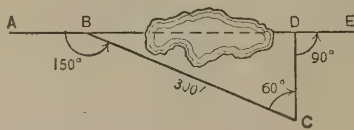


Suggestions. 1. Compare $\angle CED + \angle CDE$ with $\angle A + \angle B$. 2. Is $\angle CED = \angle CDE$? 3. Is $\angle CED = \angle A$?

Ex. 50. When surveying, it often becomes necessary to extend a straight line beyond some obstruction. Explain why EF lies on AB extended, in the adjoining figure, if the measurements are made as indicated on the figure.

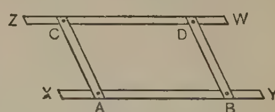


Ex. 51. Aided by the adjoining figure, tell how long a surveyor must make CD in order that DE shall lie on AB extended beyond the obstruction O, if he has already made the measurements and lines indicated in the figure.

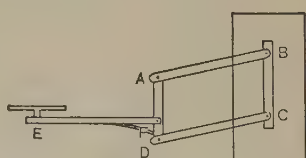


Ex. 52. Prove that two parallelograms are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.

Ex. 53. Explain why the rulers pictured in the adjoining figure can be used to draw a line parallel to the lower edge of XY, if $AB = CD$ and $AC = BD$, where A, B, C, and D are the pivots about which the rods move.

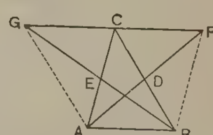


Ex. 54. Explain why the adjustable bracket shown in the adjoining figure always carries the tray at E in a horizontal position if $EF \perp AD$, $BC = AD$, $AB = CD$, and BC is vertical.



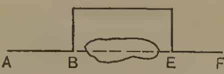
Ex. 55. If D and E are the mid-points of the sides BC and AC respectively of $\triangle ABC$, and AD be extended to F and BE to G, making $DF = AD$ and $EG = BE$, prove that GCF is a straight line and that $GC = CF$.

Suggestion. Recall § 89, page 46.



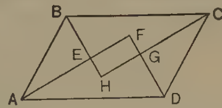
Ex. 56. When laying out the lines for the foundation of a rectangular building, as $ABCD$, contractors often measure off AD and DC at right angles and of the required lengths. Then AB is measured off equal to CD and at right angles to AD. (See figure for Ex. 181, page 70.) Why should $ABCD$ then be a rectangle?

Ex. 57. Explain why the constructions indicated in the adjoining figure give a line EF which lies on AB extended beyond the obstruction O.



Ex. 58. What angle is formed by the bisectors of two consecutive angles of: (*a*) a rectangle? (*b*) an equilateral triangle? (*c*) a parallelogram?

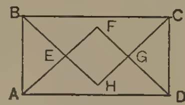
Ex. 59. Prove that the bisectors of the interior angles of a parallelogram form a rectangle.



Ex. 60. Prove that the bisectors of the angles of a rectangle form a square.

Suggestions. 1. Make a plan based upon § 143.

2. To prove $EF = EH$, prove $AF = BH$ and $AE = BE$.



Ex. 61. If on the diagonal BD of square $ABCD$ a distance BE is taken equal to AB, and if EF is drawn perpendicular to BD meeting AD at F, then $AF = EF = ED$.

Ex. 62. If the non-parallel sides of an isosceles trapezoid are extended until they meet, they form with the base an isosceles triangle.

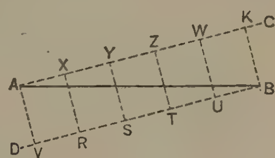
Ex. 63. The following method of dividing a segment into equal segments may be used to divide AB into five equal parts.

1. Draw AC, making with AB any convenient angle.

2. Draw BD parallel to AC.

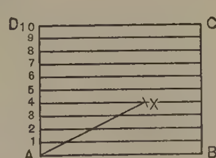
3. Lay off on AC five equal segments, and on BD five other segments of the same length.

4. Connect the points of division as in the figure.



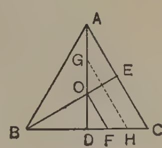
Prove now that AB is divided into five equal segments.

Ex. 64. Let $ABCD$ represent a portion of a sheet of ruled paper. Aided by the figure, tell how such a sheet of paper can be used to divide a segment greater than AD and less than AC into 2, 3, 4, 5, 6, 7, 8, 9, or 10 equal segments.



Ex. 65. If O is the point of intersection of the medians AD and BE of equilateral triangle ABC, and OF is drawn parallel to AC, meeting BC at F, prove that DF is $\frac{1}{6}$ BC.

Suggestion. Let G be the mid-point of OA, and draw $GH \parallel AC$; also imagine a line through $D \parallel AC$.



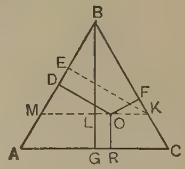
Ex. 66. If the median drawn from any vertex of a triangle is greater than, equal to, or less than, one half the opposite side, the angle at the vertex is acute, right, or obtuse respectively. (§ 151.)

Ex. 67. Prove that the median to any side of a triangle is greater than the altitude to that side unless the side is the base of an isosceles triangle.

Ex. 68. Prove that the diagonals of an oblique-angled parallelogram are unequal, the one joining the acute angles being the greater.

Ex. 69. If the base of an isosceles triangle be trisected, the lines joining the points of trisection to the vertex of the triangle are equal.

Ex. 70. Prove that the sum of the perpendiculars drawn from any point within an equilateral triangle to the sides of the triangle is equal to the altitude of the triangle.



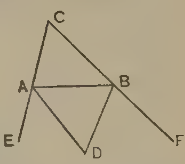
Prove $OR + OF + OD = BG$.

Suggestions. 1. Let KM be $\parallel AC$ and $KE \perp AB$.

2. Compare EK and BL.

3. Prove $OD + OF = EK$.

Ex. 71. If the exterior angles at the vertices A and B of $\triangle ABC$ are bisected by lines which meet at D, prove $\angle D = \frac{1}{2}\angle B + \frac{1}{2}\angle A$.



Proof. 1. $\angle D = 180 - \angle DAB - \angle ABD$. Why?

2. $\angle DAB = \frac{1}{2}\angle EAB = \frac{1}{2}(\angle C + \angle ABC)$. Why?

3. Similarly $\angle ABD = ?$

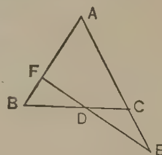
4. $180° = \angle BAC + \angle C + \angle ABC$. Why?

5. Substitute in step 1, and complete the proof.

Note. Numerical relations among angles of a figure are proved by the facts in §§ 109, 108, and 105.

Ex. 72. Prove that the exterior angle at the base of an isosceles triangle equals the angle between the bisectors of the base angles.

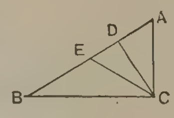
Ex. 73. D is any point in the base BC of isosceles triangle ABC. The side AC is extended from C to E, so that CE equals CD, and DE is drawn, meeting AB at F. Prove $\angle AFE = 3\angle AEF$.



Suggestions. 1. $\angle AFE$ is exterior to $\triangle BDF$.

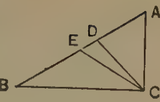
2. $\angle B = \angle ACD$, which is exterior to $\triangle CDE$.

Ex. 74. If CD is the altitude to the hypotenuse AB of right triangle ABC, and E is the mid-point of AB, prove $\angle DCE = \angle A - \angle B$.



Suggestions. 1. $\angle DCE$ is the complement of $\angle DEC$. Why?

2. Express $\angle DEC$. 3. Recall Ex. 281, page 88.



Ex. 75. If CD is the altitude to the hypotenuse AB of right triangle ABC, and CE is the bisector of $\angle C$, meeting AB at E, then $\angle DCE = \frac{1}{2}(\angle A - \angle B)$.

Suggestions. 1. $\angle DCE = \angle ACE - \angle ACD$.

2. $\angle ACE = \frac{1}{2}\,90°$.

3. $90° = \angle A + \angle B$. Why?

Ex. 76. If $\angle B$ of $\triangle ABC$ is greater than $\angle C$, and BD is drawn to AC making AD equal to AB, prove

$$\angle ADB = \tfrac{1}{2}(\angle B + \angle C), \text{ and } \angle CBD = \tfrac{1}{2}(\angle B - \angle C).$$

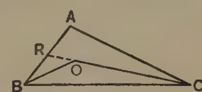
Suggestions. 1. First apply § 109.

2. $\angle DBC = \angle B - \angle ABD = \angle B - \angle ADB$. Why?

Ex. 77. Prove that the sum of any three sides of a quadrilateral is greater than the fourth side.

Suggestion. Draw a diagonal.

Ex. 78. If O is any point within $\triangle ABC$, then $BO + OC < BA + AC$.



Suggestions. 1. Extend CO until it intersects AB at R.

2. Compare BO with $BR + RO$.

3. Add OC to both members of the inequality.

4. Compare $RO + OC$ with $RA + AC$, and complete the proof.

Ex. 79. Prove that the sum of the lines drawn from any point within a triangle to the vertices is less than the sum of the three sides.

Suggestion. 1. Let O within $\triangle ABC$ be joined to A, B, and C.

2. $OA + OB < AC + BC$. (Ex. 78, above.)

3. Similarly express $OB + OC$ and also $OC + OA$.

4. Add these inequalities and divide by 2.

Ex. 80. In triangle ABC, if D is any point on AC so that $AD = AB$, then $BC > DC$.

Suggestion. Compare $BC + AB$ with AC.

Ex. 81. Prove that each of the equal sides of an isosceles triangle is greater than one half the base.

Ex. 82. If O is any point within triangle ABC, then $AO + BO + CO$ is greater than one half the perimeter.

Suggestions. 1. Apply § 149 (a) to each side of the triangle.

2. Add the inequalities and divide by 2.

Ex. 83. Prove that any side of a triangle is less than one half the perimeter of the triangle.

Suggestions. 1. Apply § 149 (a) to one side.

2. Add that side to both members of the inequality.

Ex. 84. Prove that the median to any side of a triangle is less than one half the perimeter of the triangle.

Suggestion. The median lies in each of two $\triangle$s. Apply § 149 (a) to the median in each $\triangle$, and add.

Ex. 85. Prove that the median to any side of a triangle is less than one half the sum of the other two sides of the triangle.

Suggestion. Extend the median its own length, through the side of the triangle. Connect the end of the new segment with one of the other vertices of the triangle.

Ex. 86. Prove that the median to any side of a triangle is greater than one half the sum of the other two sides diminished by the side to which it is drawn.

Ex. 87. Prove that the sum of the medians to the sides of a triangle is greater than one half the perimeter of the triangle.

Suggestion. Apply Ex. 86 to each of the medians.

Ex. 88. Define:

(*a*) parallelogram
(*b*) rectangle
(*c*) square
(*d*) rhombus
(*e*) trapezoid
(*f*) isosceles trapezoid
(*g*) altitude of a parallelogram
(*h*) altitude of a trapezoid
(*i*) median of a trapezoid

Ex. 89. (*a*) What are the important facts known about every parallelogram?

(*b*) State four theorems by which a quadrilateral can be proved to be a parallelogram.

Ex. 90. State facts known about a rectangle.

Ex. 91. State facts known about a square.

Ex. 92. State facts known about a trapezoid.

Ex. 93. State facts known about an isosceles trapezoid.

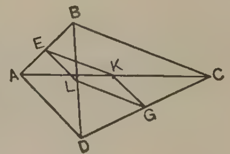
Ex. 94. State methods for proving two segments equal.

Ex. 95. State methods for proving two angles equal.

Ex. 96. State methods for proving two lines are parallel.

Ex. 97. If E and F are the mid-points of sides AB and AC respectively of $\triangle ABC$, and AD is the perpendicular from A to BC, prove $\angle EDF = \angle EAF$.

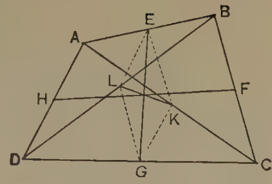
Suggestion. Recall Ex. 281, Book I.



Ex. 98. If E and G are the mid-points of AB and CD respectively of quadrilateral $ABCD$, and K and L are the mid-points of diagonals AC and BD respectively, prove that $EKGL$ is a parallelogram.

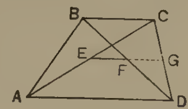
Suggestion. Recall § 159.

Ex. 99. Prove that the lines joining the mid-points of the opposite sides of a quadrilateral and the line joining the mid-points of the diagonals of the quadrilateral meet in a point.



Suggestion. Recall Ex. 98.

Ex. 100. Prove that the line joining the mid-points of the diagonals of a trapezoid is parallel to the bases and equal to $\frac{1}{2}$ their difference.

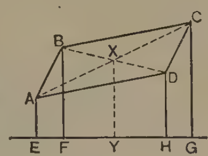


Suggestions. 1. Draw $EG \parallel AD$ meeting CD at G.

2. Prove EG passes through point F.

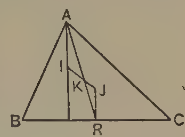
3. Compare EG with AD and FG with BC.

Ex. 101. If the perpendiculars AE, BF, CG, and DH be drawn from the vertices of parallelogram $ABCD$ to any line in its plane not intersecting its surface, prove that $AE + CG = BF + DH$.



Suggestion. See adjoining figure. Apply § 161.

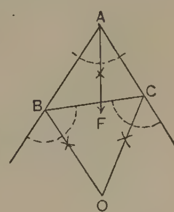
Ex. 102. Prove that the line joining the orthocenter of a triangle to the circum-center of the triangle passes through the center of gravity (§ 166) of the triangle.



Suggestions. 1. Draw AR, and try to prove that K is the center of gravity, by proving that $AK = 2\,KR$.

2. Recall § 160, and Ex. 285, Book I.

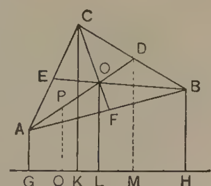
Ex. 103. The bisectors of the exterior angles at two vertices, and the bisector of the interior angle at the third vertex of a triangle are concurrent.



Suggestion. The proof is like that for § 163.

Ex. 104. If two medians of a triangle are equal, the triangle is isosceles.

Ex. 105. The perpendicular from the intersection of the medians of a triangle to any straight line in the plane of the triangle, not intersecting its surface, is equal to one third the sum of the perpendiculars from the vertices of the triangle to the same line. (§ 161.)



BOOK II

Ex. 1. The lines joining in order the ends of two intersecting diameters of a circle form a parallelogram.

Ex. 2. From point A of circle O, two chords AB and AC are drawn on opposite sides of the diameter AOD. Prove that $\angle BOC = 2\angle BAC$.

Ex. 3. (*a*) If A, B, C, and D are four consecutive points on $\odot O$, and if $\angle AOB = \angle COD$, then $\widehat{AC} = \widehat{BD}$.

(*b*) State and prove the converse of part (*a*).

Ex. 4. (*a*) If A, B, C, and D are four consecutive points on $\odot O$, and if $\angle AOC = \angle BOD$, then $\widehat{AB} = \widehat{CD}$.

(*b*) State and prove the converse of part (*a*).

Ex. 5. If the perpendiculars from a point on an arc to the radii drawn to the ends of the arc are equal, then the point bisects the arc.

Ex. 6. A quadrilateral $XYZW$ is inscribed in a circle. If XY and YZ are equal respectively to XW and WZ, then XZ is a diameter of the circle.

Ex. 7. If $XYZW$ is an inscribed quadrilateral such that $XY = ZW$, then diagonals XZ and YW also are equal.

Ex. 8. State and prove the converse of Exercise 7.

Ex. 9. In Exercise 8, prove also that $\angle XYZ = \angle YZW$.

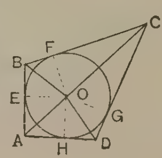
Ex. 10. If AB is one of the non-parallel sides of a trapezoid which is circumscribed about a circle O, prove $\angle AOB$ is a right angle.

Suggestion. Recall § 202, p. 112.

Ex. 11. AB and AC are the tangents to a circle from point A, and D is any point in the smaller of the arcs subtended by the chord BC. If a tangent to the circle at D meets AB at E, and AC at F, prove that the perimeter of $\triangle AEF = AB + AC$.

Ex. 12. Prove that the straight line joining the mid-points of the non-parallel sides of a circumscribed trapezoid is equal to one fourth of the perimeter of the trapezoid.

Suggestion. Recall Ex. 58, page 112.



Ex. 13. If $ABCD$ is a quadrilateral circumscribed about a circle whose center is O, prove that $\angle AOB + \angle COD = 180°$.

Suggestion. Compare $\angle EOB$ and $\angle BOF$; also $\angle EOA$ and $\angle AOH$; etc.

Ex. 14. If tangents are drawn to a circle at the extremities of any pair of diameters which are not perpendicular to each other, the figure formed is a rhombus.

Suggestion. Recall Exercises 54 and 58, page 112.

Ex. 15. If the angles of a circumscribed quadrilateral are right angles, the figure is a square.

Ex. 16. If the angles of an inscribed triangle measure 35°, 75°, and 70°, respectively, how many degrees are there in each of the arcs subtended by the sides of the triangle?

Ex. 17. If $ABCD$ is an inscribed quadrilateral; if $\angle A = 50°$; if $\angle D = 70°$; and if $\overset{\frown}{CD} = 20°$; how large are $\angle B$ and $\angle C$?

Ex. 18. The base of an inscribed isosceles triangle is the diameter of a circle. Find the size of each angle of the triangle.

Ex. 19. What kind of angle is an inscribed angle whose intercepted arc is less than a semicircle? One whose arc is more than a semicircle?

Ex. 20. How large is an angle which is inscribed in an arc of 224°?

Ex. 21. Prove Exercise 21, page 103, without drawing OC.

Ex. 22. If equal chords are drawn from opposite ends of a diameter and on opposite sides of it, they will be parallel.

Ex. 23. If two opposite sides of an inscribed quadrilateral are equal, the quadrilateral is an isosceles trapezoid.

Ex. 24. If two equal intersecting chords are drawn from the opposite ends of a diameter and on the same side of it, the line joining their other extremities is parallel to the diameter.

Ex. 25. If a diagonal of an inscribed quadrilateral bisects the angles through which it is drawn, it is a diameter of the circle.

Ex. 26. If A, B, C, and D are four points in order on a circle, and if $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD}$, then $ABCD$ is an isosceles trapezoid, whose diagonal AC bisects $\angle A$.

Ex. 27. AOB is a diameter of the circle O, and C is any point of semicircle AB. AC is extended its own length to D, and D is joined to B. Prove that $\triangle ABD$ is isosceles.

Ex. 28. Points R, S, T, W, and X, in order, divide a circle into five equal arcs. Prove that $\angle RXS = \angle SWT = \angle XSW$.

Ex. 29. If the diagonals of an inscribed quadrilateral intersect at the center of the circle, the figure is a rectangle.

Ex. 30. Prove that a parallelogram inscribed in a circle is a rectangle.

Ex. 31. Prove that the bisector of an inscribed angle bisects the arc intercepted by the angle.

Ex. 32. From a point A on a circle, a chord AB is drawn. From D, another point, chord DA is drawn and extended to C. If $\overset{\frown}{BD} = 40°$, how large is $\angle BAC$?

Ex. 33. In a figure like that for Exercise 32, prove that $\angle BAC$ is measured by $\frac{1}{2}(360° - \overset{\frown}{BD})$.

Ex. 34. $ABCD$ is a quadrilateral inscribed in a circle. Another circle is drawn upon AD, a chord, meeting AB and CD extended at E and F respectively. Prove chords BC and EF are parallel.

Suggestion. Recall Ex. 79, page 119.

Ex. 35. If the opposite angles of a quadrilateral $ABCD$ are supplementary, a circle can be circumscribed about the quadrilateral.

Suggestions. 1. Assume that D falls outside the circle through A, B, and C, and that the $\odot$ cuts CD at E.

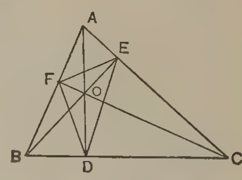
2. Derive two contradictory statements about $\angle D$ and $\angle AEC$, using the hypothesis and Ex. 79, page 119.

3. Next, assume that D falls inside the $\odot ABC$, and complete the indirect proof.

Ex. 36. A right triangle has for its hypotenuse the side of a square, and lies outside the square. Prove that the straight line drawn from the center of the square to the vertex of the right angle of the right triangle bisects the right angle.

Suggestions. The circle on the hypotenuse as diameter must pass through the center of the square and also through the vertex of the right angle of the right triangle, according to Ex. 35, above.

Ex. 37. The perpendiculars drawn from the vertices of a triangle to the opposite sides are the bisectors of the angles of the triangle formed by joining the feet of these perpendiculars.



Suggestions. 1. Ⓢ can be circumscribed about quadrilaterals $BDOF$, $CDOE$, and $AEOF$. (Ex. 35.)

2. Compare $\angle ODF$ with $\angle OBF$, and $\angle ODE$ with $\angle OCE$.

3. Compare $\angle OBF$ with $\angle OCE$, by connecting each with $\angle BAC$.

4. Then AD bisects $\angle EDF$. 5. Similarly for $\angle DEF$ and $\angle DFE$.

Ex. 38. Construct the triangle having given the feet of the perpendiculars from the vertices to the opposite sides. (Recall Ex. 37.)

Ex. 39. The angle formed by a tangent and a chord equals 55°. How large are the arcs subtended by the chord?

Ex. 40. In the figure for Case I, Prop. XV, page 113, prove that CE and ED make equal angles with AB.

Ex. 41. Prove that the tangent to a circle at the vertex of an inscribed isosceles triangle is parallel to the base of the triangle.

Ex. 42. $\triangle ABC$ is inscribed in circle O. Prove that the tangent to the circle at point C makes with AC an angle equal to $\angle B$ and that it makes with BC an angle equal to $\angle A$.

Ex. 43. Quadrilateral $ABCD$ is inscribed in a circle. Prove that $\angle ADC$ equals the angle formed by the tangent at B with AB, increased by the angle that same tangent makes with BC.

Ex. 44. Prove Proposition XIX, page 121, by drawing through B a chord parallel to CD.

Suggestion. Recall § 203.

Ex. 45. If a point is inside a circle, the lines joining it to the ends of a diameter form an obtuse angle.

Ex. 46. Points A, B, C, D, and E, in order, divide a circle into five equal arcs. BE cuts AC at F and BD cuts AC at G. How many degrees are there: (*a*) in $\angle BAC$? (*b*) in $\angle ABE$? (*c*) in $\angle AFE$? (*d*) in $\angle BFA$?

Ex. 47. In the figure for § 218, page 122, prove that $\triangle BDE$ and $\triangle BCE$ are mutually equiangular.

Ex. 48. Prove Proposition XX, page 122, by drawing through B a chord parallel to CD.

Ex. 49. If two tangents to a circle form an angle of 80°, how large are the intercepted arcs?

Ex. 50. If sides AB, BC, and CD of an inscribed quadrilateral subtend arcs of 99°, 106°, and 78° respectively, and sides BA and CD extended meet at E, and sides AD and BC at F, find the number of degrees in $\angle AED$ and $\angle AFB$.

Ex. 51. If sides AB and BC of inscribed quadrilateral $ABCD$ subtend arcs of 69° and 112° respectively, and $\angle AED$ between the diagonals is 87°, how many degrees are there in each angle of the quadrilateral?

Suggestions. 1. Let $x = \widehat{AD}$ and $y = \widehat{DC}$. Determine these arcs algebraically.

2. Then determine the size of each of the required angles.

Ex. 52. If $ABCD$ is an inscribed quadrilateral, and AD and BC extended meet at P, the tangent XY at P to the circle circumscribed about the $\triangle ABP$ is parallel to CD.

Suggestions. 1. $XY \parallel CD$ if $\angle DCP = ?$

2. Compare each of these angles with $\angle BAD$. Recall Ex. 79, page 119.

Ex. 53. The angle formed by two tangents to a circle from an external point is supplementary to the angle formed by the radii drawn to the points of contact.

Ex. 54. If AB and AC are the tangents from point A to the circle O, $\angle BAC = 2\angle OBC$.

Suggestions. 1. Draw OA. What relation does it bear to BC?
2. Compare $\angle BAO$ with $\angle OBC$.

Ex. 55. If AB and AC are tangents to a circle whose center is O from a point A, touching the circle at B and C respectively, and D is any point on the minor arc BC, then $\angle BDC = 90° + \frac{1}{2}\angle A$.

Suggestion. Extend AD to meet the $\odot$ at E. Measure $\angle BAC$ and $\angle BDC$. Eliminate $\overset{\frown}{BDC}$.

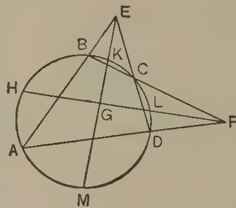
Ex. 56. If AD and AF are tangents to the circle whose center is O and E is any point in major arc DF, then $\angle DEF = 90° - \frac{1}{2}\angle A$.

Suggestions. 1. Draw OD and OF.
2. Compare $\angle E$ with $\angle DOF$, and $\angle DOF$ with $\angle A$.

Ex. 57. If $ABCD$ is a circumscribed quadrilateral, prove that the angle between the lines joining the opposite points of contact equals $\frac{1}{2}(\angle A + \angle C)$ or is supplementary to it.

Suggestion. Find the measures of $\angle A$, $\angle C$, and the $\angle$ between the two chords. Add the measures of $\angle A$ and $\angle C$.

Ex. 58. $ABCD$ is a quadrilateral inscribed in a circle. If sides AB and DC extended intersect at E, and AD and BC extended intersect at F, prove that the bisectors of $\angle E$ and $\angle F$ are perpendicular.



Suggestions. 1. $\overset{\frown}{AM} + \overset{\frown}{AH} + \overset{\frown}{KC} + \overset{\frown}{CL}$ must $= 180°$.

2. $\overset{\frown}{AM}$ enters in the measure of $\angle AEM$, and $\overset{\frown}{AH}$ in that of $\angle AFH$. Express these measures and add the results.

This will give a start on the proof.

Ex. 59. If AB and AC are the tangents to a circle from a point A, and D is any point on the major arc subtended by chord BC, prove that $\angle ABD + \angle ACD$ is constant.

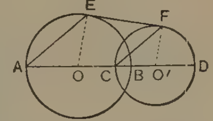
Suggestions. 1. $\angle ABD + \angle ACD = 360° - \angle A - \angle D$. Why? 2. Substitute for $\angle A$ and $\angle D$ their measures.

Ex. 60. Euclid's construction for the tangent to a circle with center M from a point A outside of it is as follows:

1. Draw the circle with center M and radius MA.
2. Draw MA intersecting the given $\odot$ at B.
3. Draw $BC \perp MA$ at B, meeting the larger $\odot$ at C.
4. Draw MC, intersecting the given $\odot$ at D.

Statement. AD is tangent to the given $\odot$.

Make the construction and give the proof.



Ex. 61. A, B, C, and D are four points in a straight line, B lying between C and D; EF is a common tangent to the circles drawn upon AB and CD as diameters. Prove $\angle BAE = \angle DCF$.

Ex. 62. Circle D is tangent internally at B to a larger circle whose center is E. If a line through B cuts circle D at C, and circle E at A, prove that $AE \parallel CD$.

Suggestion. Draw the line of centers.

Ex. 63. If a straight line be drawn through the point of contact of two circles which are tangent externally, terminating in their circumferences, the tangents at its extremities are parallel.

Suggestion. Draw the common internal tangent of the circles.

Ex. 64. If a circle be drawn upon the radius of another circle as diameter, any chord of the greater circle passing through the point of contact of the circles is bisected by the smaller circle.

Ex. 65. Construct $\triangle ABC$, having given a, h_b, and h_c.

Ex. 66. Construct $\triangle ABC$, having given A, C, and t_c.

Ex. 67. Construct $\triangle ABC$, having given c, h_c, and m_c.

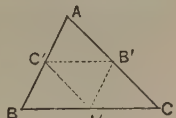
Ex. 68. Construct $\triangle ABC$, having given A, B, and h_c.

Ex. 69. Construct $\triangle ABC$, having given a, b, and A.

Ex. 70. Construct $\triangle ABC$, having given A, t_A, and h_a.

Ex. 71. Construct $\triangle ABC$, having given b, m_a, and C.

Ex. 72. Construct a right triangle, having given the altitude upon the hypotenuse and one of the legs of the triangle.

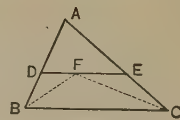


Ex. 73. Construct a triangle, having given the mid-points of its sides.

Suggestion. How does $C'B'$ compare with BC?

Ex. 74. Construct a tangent to an arc of a circle at a given point of the arc without using the center of the circle.

Ex. 75. Construct a tangent to a given circle which will be perpendicular to a given straight line.



Ex. 76. Construct a parallel to side BC of $\triangle ABC$ meeting AB and AC at D and E respectively, so that DE will equal the sum of BD and CE.

Ex. 77. Inscribe a square within a given right triangle having one of its angles coincident with the right angle of the triangle and the opposite vertex lying on the hypotenuse.

Suggestion. In the analysis figure, draw the diagonal from the vertex of the right triangle.

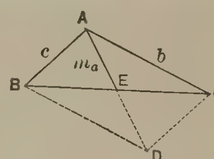
Ex. 78. Construct a rhombus within a given triangle, having one angle coincident with an angle of the triangle, and the opposite vertex lying on the opposite side of the triangle.

Ex. 79. Construct a square which will have its vertices on the sides of a given rhombus.

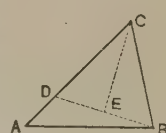
Suggestion. Make an analysis based upon the adjoining figure.

Ex. 80. Construct two tangents to a given circle which will make a given angle.

Suggestion. Draw the supplement of the given angle at the center of the circle.



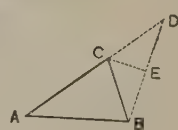
Ex. 81. Construct a $\triangle ABC$ having given c, b, and m_a. Make an analysis based upon the adjoining figure.



Ex. 82. Given the base, the smaller adjacent acute angle, and the difference between the other two sides of the triangle, construct the triangle.

Suggestions. 1. Let $CD = CB$. Then $AD = AC - BC$.

2. Then $\triangle ABD$ can be made the basis of the construction.



Ex. 83. Given the base of a triangle, the smaller adjacent angle, and the sum of the other two sides, construct the triangle.

Suggestions. 1. Let $AD = AC + CB$. Draw $CE \perp BD$.

2. $\triangle ABD$ can be made the basis of the construction.

Ex. 84. Given an angle of a triangle and the segments of the opposite side made by the altitude drawn to that side, construct the triangle.

Ex. 85. Construct a circle tangent to a given line and having its center at a given point not on the line.

Ex. 86. Construct a circle which will be tangent to each of two parallels and will pass through a given point lying between the parallels.

Ex. 87. Construct a circle having its center in a given line, and passing through two points not in the line.

Ex. 88. Construct a circle with given radius which will be tangent to a given circle and will pass through a given point outside of the circle.

Ex. 89. Construct a circle with given radius which will be tangent to a given circle and pass through a given point inside of the circle.

Ex. 90. Construct a circle with a given radius which will be tangent to a given line and also to a given circle.

Ex. 91. Construct a circle which will be tangent to a given circle at a given point on it and also tangent to a given straight line.

Ex. 92. Construct a circle which will be tangent to a given circle at a given point on it and also pass through a given point outside of the circle.

BOOK III

Ex. 1. Reduce the following ratios to simplest form:
(*a*) $15:18$; (*b*) $2m:4m$; (*c*) $x:xy$; (*d*) $(ax+ay):(2x+2y)$.

Ex. 2. Find the value of x in each of the following proportions:

$$(a)\ \frac{3}{9}=\frac{x}{20};\quad (b)\ \frac{4}{x}=\frac{5}{25};\quad (c)\ \frac{x}{7}=\frac{x-4}{5}.$$

Ex. 3. Separate a segment 20 in. long into two segments (x, and $20-x$) which have the ratio $3:2$.

Ex. 4. Separate a segment 36 in. long into two segments which have:
(*a*) the ratio $1:2$; (*b*) the ratio $4:5$; (*c*) the ratio $5:7$.

Ex. 5. In § 262, page 156, if $AD=6$, $AB=15$, $AE=5$, find EC.

Ex. 6. In § 262, page 156, if $AD=m$, $DB=r$, $AE=s$, find EC.

Ex. 7. In § 262, page 156, if $AD=10$, $AC=21$, $AE=7$, find DB.

Ex. 8. In § 262, page 156, if $AD=AE$, prove $DB=EC$, and $AB=AC$.

Ex. 9. In § 262, page 156, if $AD = 2\ AE$, and $AB = 15$, find AC.

Ex. 10. The sides of $\triangle ABC$ are $AB = 8$ in., $BC = 12$ in., and $AC = 14$ in. A line XY, parallel to BC, cuts AB at X, a point 6 in. from A, and AC at Y. How long are the segments AY and YC?

Ex. 11. The vertices of quadrilateral $ABCD$ are joined to a point O lying outside the quadrilateral. Points A', B', C', and D' are taken on OA, OB, OC, and OD respectively, so that $A'B' \parallel AB$, $B'C' \parallel BC$, and $C'D' \parallel CD$. Prove $A'D' \parallel AD$.

Ex. 12. Point O lies inside of and is joined to the vertices of pentagon $ABCDE$. Point X is placed on OA one third the way from O to A; Y on OB, one third the way from O to B; and Z, W, and K are placed similarly on OC, OD, and OE. Prove $XYZWK$ and $ABCDE$ are mutually equiangular.

Ex. 13. Let P be any point not on line AB and R any point on AB. Let S be a point in PR such that $PS : PR = 1 : 3$. Suppose that R moves along AB. What is the locus of S?

Ex. 14. XY is parallel to side AB of $\triangle OAB$, meeting OA at X and OB at Y. Point C is taken between X and A of OA, and BC is drawn. XZ is drawn parallel to BC, meeting YB at Z. Prove $CY \parallel AZ$.

Suggestion. Try to prove $OC : OA = OY : OZ$.

Ex. 15. In § 271, if $AB = 2$ in., $AC = 3$ in., and $BC = a$ in., find BD and DC.

Ex. 16. In § 271, if $AB = c$, $AC = b$, and $BC = a$, find BD and DC.

Ex. 17. In § 271, if $AC = 2\ AB$, prove $BD = \frac{1}{3} BC$.

Ex. 18. As an application of Ex. 17, show how to trisect a segment.

Ex. 19. State and prove the converse of Proposition V, § 271.

Suggestion. Extend CA to E, making $AE = AB$.

Ex. 20. Let AM be the median to side BC of $\triangle ABC$. Let MX bisect $\angle AMB$, and meet AB at X. Let MY bisect $\angle CMA$, and meet CA at Y. Prove $XY \parallel BC$.

Ex. 21. Prove that the lines joining the mid-points of the sides of a triangle form a triangle which is similar to the given triangle.

Ex. 22. Let AC be the hypotenuse of right $\triangle ABC$, and E and F be any points on AB and BC respectively; let ED and FG be perpendicular to AC, meeting AC at D and G respectively. Prove $\triangle AED \sim \triangle FGC$.

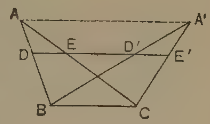
Ex. 23. The diagonals of a trapezoid whose bases are AD and BC, intersect at E. If $AE = 9$ in., $EC = 3$ in., and $BD = 16$ in., find BE and ED.

Ex. 24. AB is the hypotenuse of right $\triangle ABC$. If perpendiculars be drawn to AB at A and B, meeting AC extended at D, and BC extended at E, prove $\triangle ACE$ similar to $\triangle BCD$.

Ex. 25. If E is the mid-point of one of the parallel sides BC, of trapezoid $ABCD$, and AE and DE extended meet DC and AB extended at F and G respectively, then FG is parallel to BC.

Suggestion. Prove $GB:GA = FE:FA$.

Ex. 26. $\triangle ABC$ and $\triangle A'BC$ have their vertices A and A' in a line parallel to their common base BC. If a parallel to BC cuts AB at D, AC at E, $A'B$ at D', and $A'C$ at E', then $DE = D'E'$.



Suggestion. Prove $DE : BC = D'E' : BC$.

Ex. 27. $\triangle ABC$ is inscribed in a circle, of which AD is diameter. A tangent to the circle at D cuts AB extended at X and AC extended at Y. Prove $\triangle ABC$ similar to $\triangle AXY$.

Ex. 28. Two circles, on opposite sides of XY, are tangent to it at C. Through C, a straight line is drawn, meeting the first circle at A and the second at D; another straight line through C meets the first circle at B and the second circle at E. Prove that $AC : CD = BC : CE$.

Ex. 29. $\angle A$ of $\triangle ABC$ is a right angle. $DEFG$ is a square having E and F on BC, D on AC, and G on AB. Prove $CE : EF = EF : FB$.

Ex. 30. Point P, outside $\triangle ABC$, is joined to its vertices. Through any point X of AP, XY is drawn parallel to AB, meeting BP at Y. YZ is drawn parallel to BC, meeting PC at Z. XZ is drawn. Prove $\triangle XYZ \sim \triangle ABC$.

Ex. 31. AB and AC are the tangents to $\odot O$ from point A. If CD is drawn perpendicular to OB extended, then $AB : OB = BD : CD$.

Suggestion. Draw OA and BC. Prove $OA \perp BC$.

Ex. 32. If the height of an arch is 2 ft. and its span is 10 ft., what is the diameter of it?

Ex. 33. The center of a chord is 2 in. from the center of a circle whose diameter is 14 in. How long is the chord?

Ex. 34. Repeat Ex. 33, when the distance from the center is:

(*a*) 3 in.; (*b*) 4 in.; (*c*) 5 in.

Ex. 35. In Exercises 33 and 34, how does the length of the chord change when the distance from the center changes, and the diameter remains constant?

Ex. 36. The center of a chord which is 16 in. long is 2 in. from the center of its arc. What is the diameter of the circle?

Ex. 37. Repeat Ex. 36, when the distance from the center of the arc is: (*a*) 3 in.; (*b*) 4 in.; (*c*) 5 in.

Ex. 38. In Exercises 36 and 37, what happens to the length of the diameter if the chord remains constant and the distance of its center from the center of its arc increases?

Ex. 39. The chord of the arc of a certain circle is 40 ft. The distance from the center of the arc to the center of the chord is 4 ft. What is the diameter of the circle?

Ex. 40. A point is 2.5 in. from the center of the circle of diameter 13 in.

(*a*) What is the product of the segments of any chord through this point?

(*b*) What is the length of the shortest chord drawn through the point?

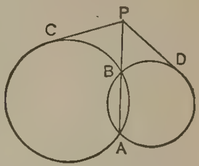
Ex. 41. Repeat Ex. 40 if the diameter is 15 in. Get the result to part (*b*) correct to one decimal place.

Ex. 42. In a circle whose diameter is 16 in., a chord 14 in. long is drawn through a point which is 4 in. from the center. What are the two segments of the chord? (Represent one segment by x.)

Ex. 43. What is the length of the tangent to a circle whose diameter is 16, from a point whose distance from the center is 17?

Ex. 44. Let r be the radius of a circle and c be the distance from the center of the circle to a point P outside the circle. Express the length of the tangent to the circle from P, in terms of r and c, using § 287.

Ex. 45. Prove that the tangents to two intersecting circles from any point in their common chord extended are equal. (Figure adjoining.)



Ex. 46. If two circles intersect, their common chord extended bisects their common tangents.

Ex. 47. The diameter which bisects a chord 12 in. long is 20 in. in length. Find the distance from one extremity of the chord to the extremities of the diameter.

Suggestions. 1. Let x represent one segment of the diameter made by the chord. 2. See § 290, page 175.

Ex. 48. A chord of a circle is 20 in. in length. Its mid-point is 5 in. from the mid-point of its arc. Find the diameter of the circle.

Ex. 49. If one leg of a right triangle is 5 in. and the hypotenuse is 10 in., what is the length of the adjacent segment of the hypotenuse made by the altitude to it?

Ex. 50. If the altitude to the hypotenuse of a right triangle is drawn, prove that the segments of the hypotenuse have the same ratio as the squares of the adjacent legs of the triangle.

Ex. 51. Construct a segment equal to $a\sqrt{2}$, where a is any segment.

Ex. 52. Construct a segment equal to $a\sqrt{5}$.

Ex. 53. Construct a segment equal to $\sqrt{a^2 - b^2}$, where a and b are any two segments.

Ex. 54. Construct a segment equal to $\frac{a}{2}\sqrt{3}$.

Ex. 55. If the bases of an isosceles trapezoid are 12 in. and 20 in. respectively, and the altitude is 10 in., how long are the non-parallel sides?

Ex. 56. If the equal sides of an isosceles trapezoid are each 6 in. long; if the upper base is 8 in., and the lower base is 14 in.; how long is the altitude?

Ex. 57. If the altitude of an equilateral triangle is 15 in., how long is a side of the triangle?

Suggestion. Represent the side by $2x$.

Ex. 58. The legs of a right triangle are 15 in. and 20 in. respectively. How long are the segments of the hypotenuse made by the bisector of the right angle?

Ex. 59. The chords drawn from a point on a circle to the ends of a diameter are 8 in. and 12 in. respectively. Find, correct to one decimal place:

(*a*) The length of the diameter of the circle.
(*b*) The distance of each of the chords from the center.
(*c*) The distance of the point from the diameter.

Ex. 60. The radius of a circle is 25 in. and the chord of one of its arcs is 14 in. (*a*) How far is the chord from the center of the circle? (*b*) How far is it from one end of the chord to each end of the diameter which is perpendicular to the chord?

Ex. 61. If the equal sides of an isosceles right triangle are each 18 in. in length, what is the length of the median drawn from the vertex of the right angle?

Ex. 62. One of the non-parallel sides of a trapezoid is perpendicular to the bases. If the length of this side is 40, and of the parallel sides 31 and 22, respectively, what is the length of the other side?

Ex. 63. If the length of the common chord of two intersecting circles is 16, and their radii are 10 and 17, what is the distance between their centers?

Ex. 64. If the diagonals of a rhombus are m and n respectively, derive a formula for the perimeter of the rhombus.

Ex. 65. The radius of a circle is 16 in. Find the length of the chord which joins the points of contact of two tangents, each 30 in. in length, drawn to the circle from a point outside the circle.

Suggestions. 1. Draw the radii to the points of contact. 2. Recall § 288.

Ex. 66. Two parallel chords on opposite sides of the center of a circle are 48 in. and 14 in. long, respectively, and the distance between their mid-points is 31 in. What is the diameter of the circle?

Suggestion. Let x represent the distance from the center to the middle point of one chord, and $31 - x$ the distance from the center to the middle point of the other. Then the square of the radius may be expressed in two ways in terms of x.

Ex. 67. The parallel sides, AD and BC, of a circumscribed isosceles trapezoid are 18 and 6 respectively. Find the diameter of the circle.

Suggestions. 1. Recall § 203, Book II; also recall Ex. 58, page 112. 2. Through B, draw $BE \parallel CD$, meeting AD at E.

Ex. 68. The diameters of two circles are 12 and 28, respectively, and the distance between their centers is 29. Find the length of the common internal tangent.

Suggestion. Find the $\perp$ drawn from the center of the smaller $\odot$ to the radius of the greater $\odot$ extended through the point of contact.

Ex. 69. Find the length of the common external tangent to two circles whose radii are 11 and 18, if the distance between their centers is 25.

Suggestion. See the figure for Exercise 238, page 149.

Ex. 70. The equal angles of an isosceles triangle are each 30°, and the equal sides are each 8 in. in length. What is the length of the base?

Suggestion. Recall Ex. 157, Book I.

Ex. 71. If D is the mid-point of leg BC of right triangle ABC, prove that the square of the hypotenuse AB exceeds 3 times the square of CD by the square of AD.

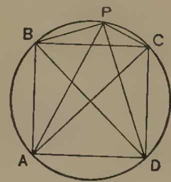
Ex. 72. If AB is the base of isosceles triangle ABC and AD is perpendicular to BC, prove $\overline{AB}^2 + \overline{BC}^2 + \overline{AC}^2 = 3\,\overline{AD}^2 + 2\,\overline{CD}^2 + \overline{BD}^2$.

Ex. 73. If D is the mid-point of leg BC of right triangle ABC, and DE is drawn perpendicular to hypotenuse AB, prove $\overline{AE}^2 - \overline{BE}^2 = \overline{AC}^2$.

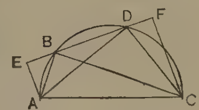
Ex. 74. If BE and CF are the medians drawn from the extremities of the hypotenuse BC of right triangle ABC, prove $4\,\overline{BE}^2 + 4\,\overline{CF}^2 = 5\,\overline{BC}^2$.

Ex. 75. If in right triangle ABC, acute angle B is double acute angle A, prove $\overline{AC}^2 = 3\,\overline{BC}^2$.

Suggestion. Recall Ex. 157, Book I.

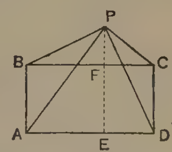


Ex. 76. Prove that the sum of the squares of the distances of any point on a circle from the vertices of an inscribed square is equal to twice the square of the diameter of the circle.



Ex. 77. If ABC and ADC are angles inscribed in a semicircle, and AE and CF are drawn perpendicular to BD extended, prove

$$\overline{BE}^2 + \overline{BF}^2 = \overline{DE}^2 + \overline{DF}^2.$$

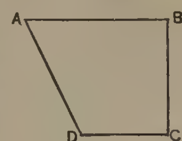


Ex. 78. If lines be drawn from any point P to the vertices of rectangle $ABCD$, prove that

$$\overline{PA}^2 + \overline{PC}^2 = \overline{PB}^2 + \overline{PD}^2.$$

Ex. 79. If two chords are perpendicular, the sum of the squares of their segments equals the square of the diameter of the circle.

Ex. 80. The sides of one pentagon are 5, 7, 8, 10, and 11 in. respectively. The shortest side of a similar pentagon is 15 in. What is the perimeter of the second pentagon?



Ex. 81. The adjoining figure is *similar* to the boundary of an irregular field of a farm; the ratio of similitude of the figure and the boundary of the field is 1 : 2400. Determine the perimeter of the field itself by first finding the perimeter of the adjoining figure and then applying § 297.

Ex. 82. The perimeter of one of two similar polygons is 153 in.; the shortest side of this polygon is 18 in. The shortest side of a similar polygon is 24 in.; what is the perimeter of the second polygon?

Ex. 83. Prove that the perimeters of two similar triangles are to each other as any two corresponding altitudes, or as any two corresponding medians, or as the bisectors of any two corresponding angles.

Ex. 84. Construct a right triangle having given its perimeter and an acute angle.

Suggestion. Any right triangle containing the given acute angle will be similar to the required triangle. The sides of the required triangle can be determined by § 297.

Ex. 85. Inscribe in a given circle a triangle similar to a given triangle.

Suggestion. Circumscribe about the given $\triangle$ a $\odot$, and draw radii to the vertices. Recall § 293.

Ex. 86. Prove that the radii of the inscribed circles of two similar triangles have the same ratio as any two corresponding sides.

Ex. 87. Two circles are on the same side of XY and tangent to it at C. CA is drawn meeting the smaller circle at B and the larger at A; CE is drawn meeting the smaller circle at D and the larger at E. Prove $CB : CA = CD : CE$.

Ex. 88. In $\triangle ABC$, altitudes AD and BE intersect at O. The perpendicular bisectors FK and HK of AC and BC respectively meet at K.

(*a*) Prove $\triangle ABO \sim \triangle FHK$.

(*b*) Prove $AO = 2\,HK$; and $BO = 2\,FK$.

Ex. 89. In Ex. 88, draw AH and OK, crossing at G. Prove $AG = 2\,GH$, and $OG = 2\,GK$.

Ex. 90. In the adjoining figure, PRS is tangent to the Ⓢ O and O'.

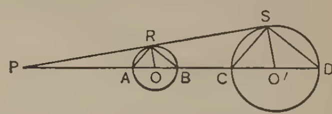
(*a*) Prove $\triangle OAR \sim \triangle CO'S$.

(*b*) Prove $\triangle ROB \sim \triangle SO'D$.

(*c*) Prove $AR \parallel CS$.

(*d*) Prove $BR \parallel DS$.

(*e*) Prove $PA \cdot PD = PB \cdot PC$.



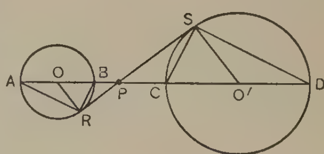
Ex. 91. In the adjoining figure, RPS is a common internal tangent to circles O and O'.

(*a*) Prove $\triangle OBR \sim \triangle O'CS$.

(*b*) Prove $BR \parallel SC$.

(*c*) Prove $AR \parallel DS$.

(*d*) Prove $PB \cdot PD = PA \cdot PC$.



Ex. 92. Prove that the bisectors of corresponding angles of two similar triangles have the same ratio as any two corresponding medians.

Ex. 93. $DEFG$ is a square having its vertices D and E on sides AB and BC respectively of triangle ABC and its vertices F and G on side AC. Let BH be $\parallel$ to AC, meeting AE extended at H; let HK be $\perp AC$ and $BT \perp AC$. Prove $BHKT$ is a square.

Ex. 94. In a given triangle, construct a square which shall have two vertices lying on one side of the triangle and having its other two vertices on the other two sides of the triangle, one on each side.

Ex. 95. $ABCD$ is a square. On AB as diameter, a semicircle is drawn lying inside the square. Center O of the circle is joined to C and D. OC cuts the semicircle at E and OD cuts it at F. From E and F perpendiculars are drawn to AB, meeting it at H and G respectively. Prove that $EFGH$ is a square.

Ex. 96. Construct a square which will have two of its vertices on a diameter of a given circle, and the remaining two vertices on the semicircle constructed on this diameter.

Ex. 97. Circumscribe about a given circle a triangle similar to a given triangle.

Suggestion. Inscribe in the given triangle a circle and draw radii to the points of tangency.

Ex. 98. If three transversals intercept proportional lengths on two parallels, the transversals meet at a point.

Suggestion. Let $A'A$ and $B'B$ meet at O and draw OC and OC'; then prove $\triangle OBC$ and $OB'C'$ similar.

Ex. 99. Prove that the non-parallel sides of a trapezoid and the line joining the middle points of the parallel sides, if extended, meet in a common point.

Ex. 100. In § 311, find b if $a = 20$ in., $c = 15$ in., and $\angle B = 60°$.

Ex. 101. Repeat Ex. 100 if $\angle B = 30°$.

Ex. 102. Repeat Ex. 100 if $\angle B = 45°$.

Ex. 103. In § 310, if $AB = 12$ in., and AB extended makes an angle of 45° with CD, find the projection of AB on CD.

Ex. 104. Repeat Ex. 103 if the angle 45° is changed to 30°.

Ex. 105. (*a*) Repeat Ex. 103 if the angle 45° is changed to 60°.

(*b*) What happens to the projection of AB on CD as the angle between AB and CD increases?

Ex. 106. Prove that the projections of two parallel sides of a parallelogram upon either of the other sides are equal.

Ex. 107. If AB and CD are equal and parallel segments, prove that p_m^{AB} equals p_m^{CD}, where m is any line.

Ex. 108. If AD and BE are the perpendiculars from vertices A and B, respectively, of acute-angled triangle ABC to the opposite sides, prove

$$AC \times AE + BC \times BD = \overline{AB}^2.$$

Suggestion. Find $2\,AC \times AE$ by § 311, and in like manner find $2\,BC \times BD$. Then add.

Ex. 109. The sides of $\triangle ABC$ are $a = 10$ in., $b = 15$ in., and $c = 20$ in. Find: (*a*) p_c^a; (*b*) p_c^b; (*c*) p_b^a; (*d*) p_a^c.

Ex. 110. Find correct to one decimal place the altitudes of the triangle whose sides are $a = 7$ in., $b = 10$ in., and $c = 15$ in.

Ex. 111. Find correct to one decimal place the altitudes of the triangle whose sides are $a = 8$ in., $b = 12$ in., and $c = 15$ in.

Ex. 112. Find the altitude to the base (b) of the triangle whose sides are a, a, and b.

Ex. 113. From the conclusion of § 312, derive a formula for p_a^b in terms of a, b, and c.

Ex. 114. In triangle ABC, if angle C equals 120°, prove

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + BC \times AC.$$

Suggestion. Recall § 312.

Ex. 115. If a line be drawn from vertex C of isosceles triangle ABC, meeting base AB extended at D, prove $\overline{CD}^2 - \overline{CB}^2 = AD \times BD$.

Suggestion. Apply § 312 in $\triangle BCD$.

Ex. 116. The sides of a triangle are $a = 6$ in., $b = 8$ in., and $c = 10$ in. Find the median to the side c.

Ex. 117. The sides of a triangle are $a = 8$ in., $b = 10$ in., and $c = 15$ in. Find the three medians of the triangle.

Ex. 118. (a) Using the conclusion of § 315 as a formula, derive from it a formula for m_c in terms of a, b, and c.

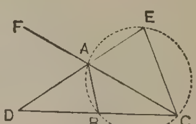
(b) Write the formula for m_a and for m_b.

Ex. 119. Find the bisectors of the angles of the triangle whose sides are $a = 3$ in., $b = 4$ in., and $c = 5$ in.

Ex. 120. Find the bisectors of the angles of the triangle whose sides are $a = 6$ in., $b = 9$ in., and $c = 12$ in.

Ex. 121. Find the bisectors of the angles of the triangle whose sides are $a = 10$ in., $b = 16$ in., and $c = 24$ in.

Ex. 122. In any triangle, the product of any two sides is equal to the product of the segments of the third side formed by the bisector of the exterior angle at the opposite vertex, minus the square of the bisector.



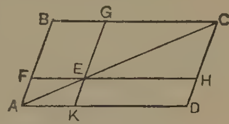
Prove $AB \times AC = DB \times DC - AD^2.$

Suggestions. 1. The solution is similar to that of § 319.
2. First prove $\triangle ABD \sim \triangle ACE$.

BOOK IV

Ex. 1. Angle B of $\triangle ABC$ is a right angle. D and E are the mid-points of AB and AC respectively. CF, perpendicular to BC at C, meets DE extended at F. Prove $\triangle ABC$ = rectangle $BCFD$.

Ex. 2. E is any point on diagonal AC of $\square ABCD$. Through E, parallels to AD and AB are drawn, meeting AB and CD at F and H respectively, and BC and AD at G and K respectively. Prove $\square FBGE = \square EHDK$.



Ex. 3. The rectangles R and S have equal altitudes.

(*a*) What part of R is S if the base of R is 8 and the base of S is 5?

(*b*) What is the ratio of R to S if the base of R is 25 and the base of S is 15?

Ex. 4. All the lots of a certain city block are rectangular and 125 ft. from front to back. Compare two lots, A and B, if their frontages are 40 ft. and 60 ft. respectively. (Do not obtain their areas.)

Ex. 5. What are the areas, correct to tenths, of the following rectangles:

(*a*) Alt. = 4.5 in.; base = 6.7 in. (*c*) Alt. = 8.4 ft.; base = 21.5 ft.
(*b*) Alt. = 3.2 yd.; base = 5.8 yd. (*d*) Alt. = 9.6 rd.; base = 25.3 rd.

Ex. 6. What is the altitude of the rectangle whose area is 336 sq. in. if its base is 32 in.?

Ex. 7. What are the dimensions of the rectangle whose area is 450 sq. in. if its length is 9 times its width?

Ex. 8. What are the dimensions of the rectangle whose area is 540 sq. ft. if the sum of the base and altitude is 47 ft.?

Ex. 9. What is the area of the parallelogram whose sides are 18 in. and 30 in. in length, if these sides include:

(*a*) an angle of 30°? (*b*) an angle of 45°? (*c*) an angle of 60°?

Ex. 10. What is the base of a triangle whose area is 480 sq. in. if its altitude is 24 in.?

Ex. 11. What is the area of the rhombus whose diagonals are 10 in. and 18 in. respectively?

Ex. 12. What is the area of the rhombus whose diagonals are m and n inches respectively?

Ex. 13. One diagonal of a rhombus is five thirds the other; the difference of the diagonals is 8 in. Determine the area of the rhombus.

Ex. 14. What is the area of the equilateral triangle whose sides are: (*a*) 16 in. long? (*b*) 20 in. long? (*c*) 15 in. long?

Ex. 15. The altitude of an equilateral triangle is 12 in. What is the area of the triangle?

Suggestion. Let $2x$ equal one of the sides.

Ex. 16. The altitude of an equilateral triangle is 3 in. Determine its area.

Ex. 17. What is the ratio of a square of side 15 in.:

(*a*) to a parallelogram whose base is 8 in. and whose altitude is 3 in.?

(*b*) to a triangle whose altitude is 5 in. and whose base is 20 in.?

Ex. 18. Divide a triangle into three equal parts by lines drawn through one of its vertices.

Ex. 19. Separate a parallelogram into three equal parts by lines drawn from one vertex.

Ex. 20. Separate a parallelogram into four equal parts by lines drawn from one vertex.

Ex. 21. Determine the area of the triangle whose sides are 25, 17, and 28.

Ex. 22. The sides of a triangular field are 10 rd., 8 rd., and 9 rd. respectively. What is the area of the field?

Ex. 23. The sides of a triangle are 14, 18, and 20. Find the area of the triangle formed by joining the mid-points of the sides.

Ex. 24. The segments of the hypotenuse of a right triangle made by the altitude to it from the opposite vertex are 4 in. and 9 in. respectively. What is the area of the triangle?

Ex. 25. What is the area of the isosceles triangle whose base is 24 in. and whose base angles are: (*a*) $45°$? (*b*) $30°$? (*c*) $60°$?

Ex. 26. What is the area of the isosceles triangle whose equal sides are each 20 in. long and whose base angles are:

(*a*) $30°$? (*b*) $45°$? (*c*) $60°$?

Ex. 27. The area of an equilateral triangle is $16\sqrt{3}$. Determine its altitude, and its side. (Use Ex. 39, page 211.)

Ex. 28. The area of an equilateral triangle is $9\sqrt{3}$. Determine its side, and its altitude.

Ex. 29. The area of an isosceles right triangle is 81 sq. in. What is the length of its hypotenuse?

Ex. 30. A circle whose diameter is 12 in. is inscribed in a quadrilateral whose perimeter is 50. What is the area of the quadrilateral?

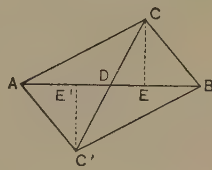
Ex. 31. Prove that the area of a polygon circumscribed about a circle is one half the product of the radius of the circle and the perimeter of the polygon.

Ex. 32. Point O lies inside $\triangle ABC$, and is joined to B and C. AO extended meets BC at Z. Prove $\triangle BOZ : \triangle COZ = \triangle ABZ : \triangle ACZ$.

Ex. 33. If D and E are the mid-points of sides BC and AC respectively of $\triangle ABC$, prove $\triangle ABD = \triangle ABE$.

Suggestion. Compare the altitudes from D and E to AB.

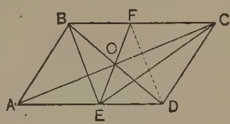
Ex. 34. Two equal triangles have a common base and lie on opposite sides of it. Prove that the base, extended if necessary, bisects the line joining their vertices. (Prove $CD = C'D$.)



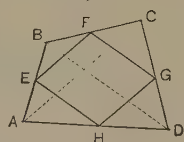
Ex. 35. If EF is any straight line drawn through the point of intersection of the diagonals of $\square ABCD$, meeting sides AD and BC at E and F respectively:

(*a*) prove $\triangle AOE = \triangle COF$;

(*b*) $\triangle ABE = \triangle FCD$; (*c*) $\triangle BOF = \triangle OED$.



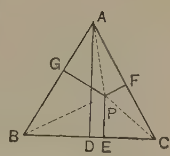
Ex. 36. If E, F, G, and H are the mid-points of sides AB, BC, CD, and DA, respectively, of quadrilateral $ABCD$, prove $EFGH$ is a parallelogram which equals one half of $ABCD$.



Ex. 37. Prove that the sum of the perpendiculars from any point inside an equilateral triangle to the three sides of the triangle is equal to the altitude of the triangle.

Suggestions. 1. $\triangle BPC + \triangle BPA + \triangle APC = \triangle ABC$.

2. Express the area of each $\triangle$ and substitute in this equation.

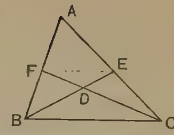


Ex. 38. If E is any point in side BC of $\square ABCD$, and DE is drawn meeting AB extended at F, prove $\triangle ABE$ equals $\triangle CEF$.

Suggestion. Compare $\triangle FCD$ with $\square ABCD$; also compare $\triangle ABE + \triangle ECD$ with $\square ABCD$.

Ex. 39. If diagonal AC of quadrilateral $ABCD$ bisects diagonal BD, then $\triangle ABC = \triangle ADC$.

Ex. 40. If D is the mid-point of side BC of $\triangle ABC$, E the mid-point of AD, F of BE, and G of CF, then $\triangle ABC = 8\,\triangle EFG$.



Ex. 41. If BE and CF are medians drawn from vertices B and C of $\triangle ABC$, intersecting at D, prove $\triangle BCD$ equals quadrilateral $AEDF$.

Suggestion. Compare $\triangle ABE$ and $\triangle BCF$ with $\triangle BEC$.

Ex. 42. Prove that two triangles are equal if two sides of one are equal to two sides of the other and the included angles are supplementary.

Suggestion. Place the triangles so that the supplementary angles are adjacent, and so that one pair of equal sides coincide.

Ex. 43. What is the area of the trapezoid whose bases are 21 in. and 35 in., respectively, and whose non-parallel side of length 20 in. makes with the lower base an angle of 30°?

Ex. 44. What is the altitude of the trapezoid whose area is 400 sq. in., and whose bases are 35 in. and 45 in. respectively?

Ex. 45. What is the upper base of the trapezoid whose area is 660 sq. in., if the lower base is 50 in. and the altitude is 20 in.?

Ex. 46. What is the side of the square whose area equals that of the isosceles trapezoid whose bases are 18 in. and 12 in., respectively, and whose equal sides are 6 in.?

Ex. 47. What is the side of the square whose area equals that of:

(*a*) a parallelogram whose dimensions are 12 in. and 36 in.?

(*b*) a triangle whose dimensions are 18 in. and 30 in.?

(*c*) a trapezoid whose bases are 15 in. and 35 in., and whose altitude is 20 in.?

Ex. 48. Prove that the line joining any vertex of a parallelogram to the mid-point of one of the non-adjacent sides separates the parallelogram into a triangle and a trapezoid such that the trapezoid is 3 times as large as the triangle.

Ex. 49. If in § 335, p. 214, the area of $\triangle A'B'C' = 147$ sq. in., $AB = 9$ in., and $A'B' = 3$ in., what is the area of $\triangle ABC$?

Ex. 50. If in the figure for § 335, $AB = 9$ in., $A'B' = 7$ in., and the area of $\triangle A'B'C' = 147$ sq. in., what is the area of $\triangle ABC$?

Ex. 51. What is the ratio of the areas of two equilateral triangles whose sides are 8 in. and 6 in. respectively?

Ex. 52. How long must the sides of an equilateral triangle be in order that its area shall be double that of an equilateral triangle whose sides are 10 in. long?

Ex. 53. What is the ratio of the areas of two similar triangles whose corresponding sides are 3 in. and 4 in. long respectively?

Ex. 54. Two triangles are similar and the second is 4 times as large as the first. If one side of the first is 5 in. long, how long is the corresponding side of the second?

Ex. 55. Two triangles are similar, and a side of one is 6 in. long. How long is the corresponding side of the other, if the second is 3 times as large as the first?

Ex. 56. The area of a certain triangle is $\frac{8}{9}$ the area of a similar triangle. If the altitude of the first is 4 in., what is the corresponding altitude of the second?

Ex. 57. What is the ratio of the areas of two isosceles triangles whose vertex angles are 50°, if the sides of the one are 10 in. long and the sides of the other are 15 in. long?

Ex. 58. The base and altitude of a certain triangle are 15 in. and 25 in. respectively. 5 in. from the vertex of the triangle, a line is drawn parallel to the base. What is:

(*a*) the area of the triangle above this line?

(*b*) the area of the trapezoid below the line?

Ex. 59. The sides AB and AC of $\triangle ABC$ are 14 in. and 22 in., respectively. From a point D in AB, a parallel to BC is drawn meeting AC at E and dividing the triangle into two equal parts. Find AD and AE.

Ex. 60. The base of a given triangle is 24 in. How far from the vertex must a line be drawn parallel to the base so that the part above the line shall be one fourth of the whole triangle?

Ex. 61. Repeat Ex. 60, changing one fourth to two thirds.

Ex. 62. Prove that the areas of two similar triangles are to each other as the squares of the bisectors of any two corresponding angles.

Ex. 63. Prove that the areas of two similar triangles are to each other as the squares of any two corresponding medians.

Ex. 64. Prove that the areas of two similar triangles are to each other as the squares of their perimeters.

Ex. 65. The sides of a triangle are 8, 10, and 12. What are the sides of a similar triangle whose area is 4 times as large as that of the first triangle?

Ex. 66. Repeat Ex. 65 if the second triangle is twice as large as the first triangle.

Ex. 67. What is the perimeter of an equilateral triangle which is 9 times as large as the equilateral triangle whose sides are 4 in. long?

Ex. 68. The areas of two similar polygons are 50 sq. in. and 450 sq. in. respectively. A side of the first is 2 in. long. How long is the corresponding side of the second?

Ex. 69. The area of one polygon is 4 times that of a similar polygon. How does any side of the first compare with the corresponding side of the second?

Ex. 70. If the area of a polygon, one of whose sides is 15 in., is 375 sq. in., what is the area of a similar polygon if the corresponding side is 10 in. long?

Ex. 71. If the area of a polygon, one of whose sides is 36 ft., is 648 sq. ft., what is the corresponding side of a similar polygon whose area is 392 sq. ft.?

Ex. 72. Prove that the areas of two similar polygons are to each other as the squares of their perimeters.

Ex. 73. Prove that the areas of two similar polygons are to each other as the squares of any two corresponding diagonals.

Ex. 74. The area of one polygon is 350 sq. in., and one of its diagonals is 15 in. The corresponding diagonal of a similar polygon is 20 in. What is the area of the second polygon?

Ex. 75. Two sides of a triangle are 8 in. and 12 in. respectively. The altitude to the third side is 5 in., what is the area?

Ex. 76. If the longer diagonal of a rhombus is 40 in., and the side of the rhombus is 30 in., what is the area of the rhombus?

Ex. 77. If b is the base and s is one of the equal sides of an isosceles triangle, prove that the area is $\frac{1}{4} b\sqrt{4s^2 - b^2}$.

Ex. 78. If the longer diagonal of a rhombus is $2d$ in. and one of the sides is s in., what is the area of the rhombus?

Ex. 79. Construct a triangle equal to the sum of two given similar triangles and similar to them.

Ex. 80. Construct an isosceles triangle equal to the sum of and similar to each of two isosceles triangles whose vertex angles are a given angle.

Ex. 81. Two similar triangles have corresponding sides of 8 in. and 15 in. respectively. How long is the side of a triangle which is similar to them and equal to their sum?

Ex. 82. Construct an isosceles triangle equal to the sum of two isosceles triangles if the base angles of all the triangles are to equal a given angle.

BOOK V

Ex. 1. Write a formula for the exterior angle at the vertex of a regular polygon of n sides.

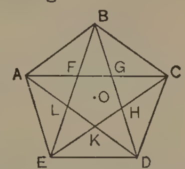
Ex. 2. Prove that the central angle of a regular polygon is the supplement of the vertex angle of the polygon.

Ex. 3. Prove that the diagonal AC of regular polygon $ABCDE$ is parallel to side DE. (Circumscribe a circle about $ABCDE$.)

Ex. 4. If $ABCDE$ is a regular polygon inscribed in circle O, prove that the diameter perpendicular to DE passes through B, and is also the perpendicular bisector of diagonal AC.

Ex. 5. If the diagonals AC and BE of a regular inscribed pentagon $ABCDE$ intersect at F, prove that $\triangle ABF$ is isosceles.

Ex. 6. If the diagonals AC and BE of regular pentagon $ABCDE$ intersect at F, prove that $BE = AE + AF$.



Ex. 7. Prove that the figure $FGHKL$, formed by parts of the diagonals of regular pentagon $ABCDE$, is also a regular pentagon.

Ex. 8. If a circle has a radius of r in., compute the side, perimeter, apothem, and area of the square circumscribed about it.

Ex. 9. What is the relation between the areas of the inscribed and circumscribed squares of a circle?

Ex. 10. Prove that the opposite sides of a regular octagon are parallel.

Ex. 11. Prove that diagonals HC and DG of regular octagon $ABCDEFGH$ are parallel.

Ex. 12. Prove that figure $ACDF$ of regular octagon $ABCDEFGH$ is an isosceles trapezoid.

Ex. 13. Prove that diagonal AE of regular octagon $ABCDEFGH$ is the perpendicular bisector of diagonal BH.

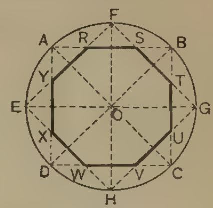
Ex. 14. A square and a regular octagon are inscribed in a circle. Prove that: (*a*) a side of the octagon is less than a side of the square;

(*b*) the perimeter of the octagon is greater than the perimeter of the square;

(Continued on page 306.)

(*c*) the apothem of the octagon is greater than the apothem of the square;

(*d*) the area of the octagon is greater than the area of the square.



Ex. 15. Construct a regular octagon whose sides shall be 1 in.

Ex. 16. Prove that the construction indicated in the adjoining figure serves to inscribe a regular octagon in the square.

Ex. 17. Prove that the sum of the perpendiculars drawn to the sides of any regular polygon from a point inside the polygon equals the apothem multiplied by the number of sides of the polygon.

Suggestions. Connect the point with each of the vertices of the polygon. Observe that the sum of the areas of the resulting triangles equals the area of the polygon. Express these areas, and form an equation.

Ex. 18. Compute the side, perimeter, apothem, and area of the regular hexagon inscribed in a circle of radius 8 in.

Ex. 19. Repeat Ex. 18 when the radius is r.

Ex. 20. Compute the side, perimeter, apothem, and area of the equilateral triangle inscribed in a circle of radius 6 in.

Ex. 21. Repeat Ex. 20 when the radius is r.

Ex. 22. Compute the side, perimeter, apothem, and area of the equilateral triangle circumscribed about the circle of radius 4.

Ex. 23. Repeat Ex. 22 when the radius is r.

Ex. 24. Compute the side, perimeter, apothem, and area of the regular hexagon circumscribed about the circle of radius 10.

Ex. 25. Repeat Ex. 24 when the radius is r.

Ex. 26. What is the relation between the perimeters of the inscribed and the circumscribed equilateral triangles of a given circle?

Ex. 27. What is the relation between the areas of the inscribed and the circumscribed equilateral triangles of a given circle?

Ex. 28. The altitude of an equilateral triangle is 8 in.

(*a*) How long is a side of the triangle?

(*b*) What is the area of the triangle?

Ex. 29. Repeat Ex. 28, when the length of the altitude is a.

Ex. 30. The apothem of a regular hexagon is 6 in.

(*a*) How long is a side of the hexagon?

(*b*) What is the area of the hexagon?

Ex. 31. Repeat Ex. 30 when the apothem is a in. long.

Ex. 32. Prove that the altitude of the equilateral triangle circumscribed about a circle of radius r is $3r$.

Ex. 33. In a given equilateral triangle inscribe a regular hexagon having two of its vertices lying on each side of the triangle. Compare the perimeter and area of the resulting hexagon with those of the given triangle.

Ex. 34. Compare the circumferences of two circles if a radius of the second is twice that of the first; if it is three times that of the first.

Ex. 35. The side of a square is 8 in. What are the circumferences of the inscribed and circumscribed circles?

Ex. 36. On a circle of radius 10 in., how long is the arc subtended by the side of an inscribed:

(*a*) regular hexagon;
(*b*) equilateral triangle;
(*c*) square;
(*d*) regular octagon?

Ex. 37. Construct a circle whose length equals the sum of the lengths of two given circles.

Ex. 38. If the length of a quadrant is 1, what is the radius of the circle?

Ex. 39. The perimeter of the regular hexagon circumscribed about a circle is $12\sqrt{3}$. What is the circumference of the circle?

Ex. 40. If quadrilateral $ABCD$ is circumscribed about a circle of radius 8 in.; if $\angle A = 60°$, $\angle B = 100°$, $\angle C = 110°$, and $\angle D = 90°$; how long is each of the arcs between the points of contact?

Ex. 41. Draw three concentric circles whose circumferences are to each other as $1:2:3$.

Ex. 42. What is the length of the boundary of the circular segment between a side of an inscribed square in a circle of radius 5 in. and the minor arc subtended by that side?

Ex. 43. Repeat Ex. 42 when the radius is r.

Ex. 44. Repeat Ex. 42 if the chord is the side of a regular inscribed hexagon.

Ex. 45. Repeat Ex. 42 if the chord is the side of a regular inscribed triangle.

Ex. 46. Compare the areas of two circles if the radius of the second is twice that of the first; if the radius of the second is 3 times that of the first.

Ex. 47. Find the radius and the circumference of the circle whose area is 64π sq. in.; 81π sq. in.; 225π sq. in.

Ex. 48. Find the side of the square equal to the circle whose diameter is 14 in.

Ex. 49. What is the area of the circle inscribed in the square whose side is 8 in.?

Ex. 50. What is the area of the circle inscribed in the regular hexagon whose side is 6 in.?

Ex. 51. What is the area of the circle inscribed in the equilateral triangle whose side is 12 in.?

Ex. 52. What is the area of the circle circumscribed about the square whose side is 10 in.?

Ex. 53. What is the area of the circle circumscribed about the regular hexagon whose side is 4 in.?

Ex. 54. What is the area of the circle circumscribed about the equilateral triangle whose side is 12 in.?

Ex. 55. How much larger is the area of the cross section of a gas pipe 8 in. in diameter than one which is 6 in. in diameter?

Ex. 56. What is the radius of the circle whose area equals that of a regular hexagon whose side is 9 in.?

Ex. 57. What is the radius of the circle whose area equals that of the trapezoid whose bases are 15 in. and 25 in. respectively, and whose altitude is 12 in.?

Ex. 58. What is the area of the quadrant of the circle whose radius is 7 in.?

Ex. 59. Express in terms of the radius R the area of the segment of the circle lying between a side of the regular inscribed hexagon and the minor arc subtended by this side.

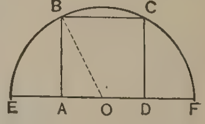
Ex. 60. Construct a circle whose area equals the sum of the areas of two given circles.

Ex. 61. Find the radius of the circle whose area is one half the area of the circle whose radius is 15.

Ex. 62. If the radius of a circle is $3\sqrt{3}$, what is the area of the sector whose central angle is 150°?

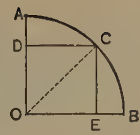
Ex. 63. The area of a regular hexagon inscribed in a circle is $24\sqrt{3}$. What is the area of the circle?

Ex. 64. Prove that the square inscribed in a semi-circle is equal to two fifths the square inscribed in the entire circle.

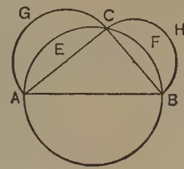


Suggestions. Let R = the radius of the circle. Compute the areas of the two squares.

Ex. 65. Prove that the area of the square inscribed in a sector whose central angle is a right angle is equal to one half the square on the radius.



Ex. 66. If a circle is circumscribed about a right triangle, and on each of the legs of the triangle as diameters semicircles are drawn, exterior to the triangle, the sum of the areas of the crescents thus formed equals the area of the triangle.



Prove area $AECG$ + area $BFCH$ = area $\triangle ABC$.

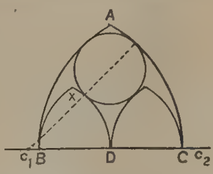
Suggestion. From the sum of $\triangle ABC$ and the semicircles on AC and BC, subtract the semicircle on AB. Express each area in terms of sides a, b, and c of the triangle.

Ex. 67. Construct three equal circles having the vertices of an equilateral triangle as their centers and for their radii one half the side of the triangle. Compute the area of that part of the interior of the triangle which is exterior to each of the circles, if the length of the side of the triangle is s.

Ex. 68. Upon a segment AC draw a semicircle. Upon AC locate a point B, not the center of AC. Upon AB and BC as diameters draw semicircles within the one drawn upon AC as diameter. Prove that the area of the surface lying within the largest semicircle and exterior to the smaller ones equals the area of the circle drawn upon BD as diameter, where BD is the perpendicular to AC at B meeting the largest semicircle at D. (Due to Archimedes.)

Ex. 69. With the vertices of an equilateral triangle as centers and the side of the triangle as radius, three equal circles are drawn. Determine the area of that figure which is common to the three circles.

Ex. 70. The arch ABC is a *lancet* arch. It consists of two arcs with equal radii, drawn from centers C_1 and C_2 outside the span BC. Within the arch are two other lancet arches.



Let $BC = 2a$; let $C_1C_2 = s$; let $BD = DC$.

(*a*) Determine the height h of the arch BAC.

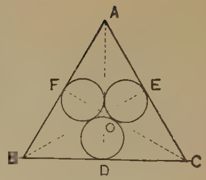
(*b*) What is the length of the radius of the arc XD?

(*c*) What is the height of the arch BXD?

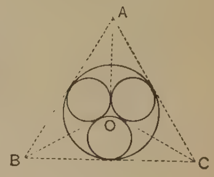
(*d*) What is the radius of the circle indicated as tangent to the arches?

(*e*) What is the area and circumference of the circle?

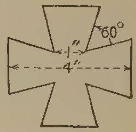
Ex. 74. In a given equilateral triangle, inscribe three equal circles, tangent to each other and each tangent to one and only one side of the triangle.



Ex. 75. In a given circle, inscribe three equal circles, tangent to each other and to the given circle.



Ex. 76. Construct a Maltese cross having the dimensions indicated.



Ex. 77. Construct a figure like Fig. 1, below, upon a square of side 2 in.

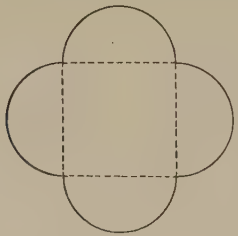


Fig. 1

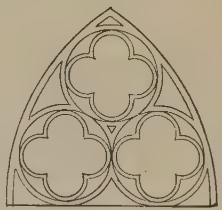


Fig. 2

(*a*) What is the length of the curved line when the side of the square is *s* inches?

(*b*) What is the total area within the curved line when the side of the square is *s* inches?

(*c*) Notice that the curved line of Fig. 1 is the fundamental unit of the adjoining window design.

Illustrative Examples in Arithmetic and Algebra

I. *To add or subtract fractions, change them to equivalent fractions having their lowest common denominator.*

1.

$$\frac{1}{6}+\frac{5}{8}$$
$$=\frac{4}{24}+\frac{15}{24}$$
$$=\frac{19}{24}$$

2.

$$\frac{3}{2x}-\frac{5}{3x}$$
$$=\frac{9}{6x}-\frac{10}{6x}$$
$$=-\frac{1}{6x}$$

3.

$$\frac{2-x}{3}-\frac{1+x}{2}$$
$$=\frac{4-2x}{6}-\frac{3+3x}{6}$$
$$=\frac{(4-2x)-(3+3x)}{6}$$
$$=\frac{4-2x-3-3x}{6}$$
$$=\frac{1-5x}{6}$$

II. *To express a quotient or a product, correct to tenths.*

1. Divide 23.486 by 2.25 correct to tenths.

```
          10.43
2‸25.)23‸48.60
      22 5
         98 6
         90 0
          8 60
          6 75
```

The quotient is 10.43, or 10.4.

2. Find 2.4×3.1416, correct to tenths.

```
 3.1416
    2.4
 125664
 62832
7.53984
```

The product is 7.5.

III. *To find a square root correct to tenths.*

1. Find $\sqrt{456.8}$ to tenths.

$$\begin{array}{rr|l}
 & & \;\;2\;\;\;1.\;3\;\;\;7 \\
\sqrt{4} = 2 & & \overline{4\;\;56.80\;\;00} \\
2^2 = & & 4 \\ \hline
2 \times 20 = & 40 & 56 \\
56 \div 40 = & \underline{1} & \\
\text{Add} & 41 & \\
1 \times 41 = & & 41 \\ \hline
2 \times 210 = & 420 & 15\;\;80 \\
1580 \div 420 = & \underline{3} & \\
\text{Add} & 423 & \\
3 \times 423 = & & 12\;\;69 \\ \hline
2 \times 2130 = & 4260 & 3\;\;11\;\;00 \\
31100 \div 4260 = & \underline{7} & \\
\text{Add} & 4267 & \\
7 \times 4267 = & & 2\;\;98\;\;69
\end{array}$$

$\therefore$ the square root correct to tenths is 21.4.

2. Find $\sqrt{21 \times 15 \times 10}$ to tenths.

$$\begin{aligned}
\sqrt{21 \times 15 \times 10} &= \sqrt{3 \times 7 \times 3 \times 5 \times 2 \times 5} \\
&= \sqrt{3^2 \times 7 \times 2 \times 5^2} \\
&= 3 \times 5\sqrt{14} \\
&= 15 \times 3.741 \\
&= 56.115, \text{ or } 56.1.
\end{aligned}$$

3. Find $(3 - \sqrt{2})^2$ correct to tenths.

$$\begin{aligned}
&(3 - \sqrt{2})^2 \\
&= 3^2 - 2 \times 3 \times \sqrt{2} + (\sqrt{2})^2 \\
&= 9 - 6\sqrt{2} + 2 \\
&= 11 - 6\sqrt{2} \\
&= 11 - 6 \times 1.414 \\
&= 11 - 8.484 \\
&= 2.516
\end{aligned}$$

$\therefore (3 - \sqrt{2})^2 = 2.5.$

4. Find $\sqrt{\frac{3}{2}}$ correct to tenths.

$$\begin{aligned}
\sqrt{\tfrac{3}{2}} &= \sqrt{\tfrac{6}{4}} \\
&= \frac{\sqrt{6}}{2} \\
&= \frac{2.449}{2} \\
&= 1.224
\end{aligned}$$

$\therefore \sqrt{\frac{3}{2}} = 1.2.$

IV. *Solving equations.*

Let A_2 mean "add 2 to both sides of the previous equation."

Let S_3 mean "subtract 3 from both sides of the previous equation."

Let M_4 mean "multiply both sides of the previous equation by 4."

Let D_2 mean "divide both sides of the previous equation by 2."

1. Solve for x:

$$\begin{array}{lrl} & 3x - a &= b + x \\ A_a & 3x &= a + b + x \\ S_x & 2x &= a + b \\ D_2 & x &= \dfrac{a+b}{2}. \end{array}$$

2. Solve for x:

$$\begin{array}{lrl} & 25 &= \frac{1}{2}(60 - x) \\ M_2 & 50 &= 60 - x \\ A_x & 50 + x &= 60 \\ S_{50} & x &= 10. \end{array}$$

3. Solve for x:

$$\begin{array}{lrl} & 2x + \angle A &= 90 \\ S_{\angle A} & 2x &= 90 - \angle A \\ D_2 & x &= \dfrac{90 - \angle A}{2} \end{array}$$

4. Solve for x:

$$\begin{array}{lrl} & (2x)^2 &= x^2 + 8^2 \\ & 4x^2 &= x^2 + 64 \\ S_{x^2} & 3x^2 &= 64 \\ D_3 & x^2 &= \frac{64}{3} \\ & x &= \dfrac{8}{\sqrt{3}} \\ & x &= \dfrac{8}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} \\ & x &= \dfrac{8\sqrt{3}}{3} \\ & x &= \dfrac{8 \times 1.732}{3} \\ & x &= \dfrac{13.856}{3}, \text{ or } 4.6 \end{array}$$

5. Solve for b:

$$\begin{array}{lrl} & \dfrac{b^2}{4}\sqrt{3} &= 24 \\ M_4 & b^2\sqrt{3} &= 96 \\ D_{\sqrt{3}} & b^2 &= \dfrac{96}{\sqrt{3}} \\ & \therefore\ b^2 &= \dfrac{96\sqrt{3}}{3} \\ & b^2 &= 32\sqrt{3} \\ & b^2 &= 32 \times 1.732 \\ & b^2 &= 55.424 \\ & b &= \sqrt{55.424} \\ & b &= 7.44, \text{ or } 7.4 \end{array}$$

6. Solve for r if $\pi = 3\frac{1}{7}$:

$$\begin{array}{lrl} & \pi r^2 &= 250 \\ & \frac{22}{7} r^2 &= 250 \\ M_7 & 22 r^2 &= 1750 \\ D_{32} & r^2 &= 79.54 \\ & r &= 79.54 \\ & r &= 8.91, \text{ or } 8.9 \end{array}$$

7. Solve for x:

$$x^2 - 6x - 27 = 0$$
$$\therefore (x-9)(x+3) = 0$$
$$\therefore x = 9; \; x = -3.$$

8. Solve for x, by completing the square.

$$2x^2 - 6x - 5 = 0$$

D_2 $$x^2 - 3x - \tfrac{5}{2} = 0$$

$A_{\frac{5}{2}}$ $$x^2 - 3x = \tfrac{5}{2}$$

$$\tfrac{1}{2} \text{ of } (-3) = -\tfrac{3}{2}; \; (-\tfrac{3}{2})^2 = \tfrac{9}{4}$$

$A_{\frac{9}{4}}$ $$x^2 - 3x + \tfrac{9}{4} = \tfrac{5}{2} + \tfrac{9}{4}$$

$$\therefore (x - \tfrac{3}{2})^2 = \tfrac{19}{4}$$

$$\therefore x - \frac{3}{2} = \frac{\sqrt{19}}{2} \qquad \therefore x_1 = \frac{7.358}{2}$$

$$\therefore x = \frac{3}{2} \pm \frac{\sqrt{19}}{2} \qquad x_1 = 3.679$$

$$\text{or } x = 3.7$$

$$\therefore x = \frac{3 \pm \sqrt{19}}{2} \qquad x_2 = \frac{-1.358}{2}$$

$$\therefore x = \frac{3 \pm 4.358}{2} \qquad x_2 = -.679$$

$$\text{or } x_2 = -.7$$

9. Solve for x by the formula. $2x^2 - 6x - 5 = 0$.

The formula is: $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 2; \; b = -6; \; c = -5.$$

$$\therefore x = \frac{+6 \pm \sqrt{36 - 4 \times 2 \times (-5)}}{2 \times 2}$$

$$\therefore x = \frac{6 \pm \sqrt{36 + 40}}{4}$$

$$\therefore x = \frac{6 \pm \sqrt{76}}{4}$$

$$\therefore x = \frac{6 \pm 8.71}{4}$$

$$\therefore x_1 = \frac{14.71}{4} \qquad x_2 = \frac{-2.71}{4}$$

$$= 3.67, \text{ or } 3.7 \qquad = -.67, \text{ or } -.7$$

I. Formula for the altitude of a triangle whose sides are a, b, and c, and whose perimeter is $2s$:

$$\text{altitude to side } a = \quad h_a = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}.$$

The altitude of an equilateral triangle whose side is b:

$$h_b = \frac{b}{2}\sqrt{3}.$$

II. Formulas for areas of plane figures.

Let a = the number of linear units in the altitude;
b = the number of the same kind of units in the base;
A = the number of the corresponding surface units in the area.
Let r = the number of linear units in the radius.
Let p = the number of linear units in the perimeter.

For the Area of	The Formula is
1. A rectangle.	1. $A = ab$.
2. A parallelogram.	2. $A = ab$.
3. A triangle.	3. $A = \frac{1}{2}ab$.
4. A trapezoid.	4. $A = \frac{1}{2}a(b_1 + b_2)$.
5. A circle.	5. $A = \pi r^2$, or $A = \frac{1}{4}\pi d^2$.
6. A regular polygon.	6. $A = \frac{1}{2}$ apothem $\times p$.
7. An equilateral triangle.	7. $A = \frac{b^2}{4}\sqrt{3}$, where b = one side.
8. Any triangle.	8. $A = \sqrt{s(s-a)(s-b)(s-c)}$.
9. A sector of a circle.	9. $A = \frac{n}{360} \times \pi r^2$, where n = the number of degrees in the $\angle$ of the sector.

III. 1. The circumference of a circle $= 2\pi r$; or πd.

2. The length of an arc of a circle $= \frac{n}{360} \times 2\pi r$, where n = the number of degrees in the arc.

No.	Sqs.	Sq. Roots	Cubes	Cube Roots	No.	Sqs.	Sq. Roots	Cubes	Cube Roots
1	1	1.000	1	1.000	**51**	2,601	7.141	132,651	3.708
2	4	1.414	8	1.259	**52**	2,704	7.211	140,608	3.732
3	9	1.732	27	1.442	**53**	2,809	7.280	148,877	3.756
4	16	2.000	64	1.587	**54**	2,916	7.348	157,464	3.779
5	25	2.236	125	1.709	**55**	3,025	7.416	166,375	3.802
6	36	2.449	216	1.817	**56**	3,136	7.483	175,616	3.825
7	49	2.645	343	1.912	**57**	3,249	7.549	185,193	3.848
8	64	2.828	512	2.000	**58**	3,364	7.615	195,112	3.870
9	81	3.000	729	2.080	**59**	3,481	7.681	205,379	3.893
10	100	3.162	1,000	2.154	**60**	3,600	7.745	216,000	3.914
11	121	3.316	1,331	2.223	**61**	3,721	7.810	226,981	3.936
12	144	3.464	1,728	2.289	**62**	3,844	7.874	238,328	3.957
13	169	3.605	2,197	2.351	**63**	3,969	7.937	250,047	3.979
14	196	3.741	2,744	2.410	**64**	4,096	8.000	262,144	4.000
15	225	3.872	3,375	2.466	**65**	4,225	8.062	274,625	4.020
16	256	4.000	4,096	2.519	**66**	4,356	8.124	287,496	4.041
17	289	4.123	4,913	2.571	**67**	4,489	8.185	300,763	4.061
18	324	4.242	5,832	2.620	**68**	4,624	8.246	314,432	4.081
19	361	4.358	6,859	2.668	**69**	4,761	8.306	328,509	4.101
20	400	4.472	8,000	2.714	**70**	4,900	8.366	343,000	4.121
21	441	4.582	9,261	2.758	**71**	5,041	8.426	357,911	4.140
22	484	4.690	10,648	2.802	**72**	5,184	8.485	373,248	4.160
23	529	4.795	12,167	2.843	**73**	5,329	8.544	389,017	4.179
24	576	4.898	13,824	2.884	**74**	5,476	8.602	405,224	4.198
25	625	5.000	15,625	2.924	**75**	5,625	8.660	421,875	4.217
26	676	5.099	17,576	2.962	**76**	5,776	8.717	438,976	4.235
27	729	5.196	19,683	3.000	**77**	5,929	8.774	456,533	4.254
28	784	5.291	21,952	3.036	**78**	6,084	8.831	474,552	4.272
29	841	5.385	24,389	3.072	**79**	6,241	8.888	493,039	4.290
30	900	5.477	27,000	3.107	**80**	6,400	8.944	512,000	4.308
31	961	5.567	29,791	3.141	**81**	6,561	9.000	531,441	4.326
32	1,024	5.656	32,768	3.174	**82**	6,724	9.055	551,368	4.344
33	1,089	5.744	35,937	3.207	**83**	6,889	9.110	571,787	4.362
34	1,156	5.830	39,304	3.239	**84**	7,056	9.165	592,704	4.379
35	1,225	5.916	42,875	3.271	**85**	7,225	9.219	614,125	4.396
36	1,296	6.000	46,656	3.301	**86**	7,396	9.273	636,056	4.414
37	1,369	6.082	50,653	3.332	**87**	7,569	9.327	658,503	4.431
38	1,444	6.164	54,872	3.361	**88**	7,744	9.380	681,472	4.447
39	1,521	6.245	59,319	3.391	**89**	7,921	9.433	704,969	4.464
40	1,600	6.324	64,000	3.419	**90**	8,100	9.486	729,000	4.481
41	1,681	6.403	68,921	3.448	**91**	8,281	9.539	753,571	4.497
42	1,764	6.480	74,088	3.476	**92**	8,464	9.591	778,688	4.514
43	1,849	6.557	79,507	3.503	**93**	8,649	9.643	804,357	4.530
44	1,936	6.633	85,184	3.530	**94**	8,836	9.695	830,584	4.546
45	2,025	6.708	91,125	3.556	**95**	9,025	9.746	857,375	4.562
46	2,116	6.782	97,336	3.583	**96**	9,216	9.797	884,736	4.578
47	2,209	6.855	103,823	3.608	**97**	9,409	9.848	912,673	4.594
48	2,304	6.928	110,592	3.634	**98**	9,604	9.899	941,192	4.610
49	2,401	7.000	117,649	3.659	**99**	9,801	9.949	970,299	4.626
50	2,500	7.071	125,000	3.684	**100**	10,000	10.000	1,000,000	4.641

FOUR-PLACE TABLE OF VALUES OF TRIGONOMETRIC RATIOS

ANGLE	SIN	COS	TAN	ANGLE	SIN	COS	TAN
1°	.0175	.9998	.0175	46°	.7193	.6947	1.0355
2°	.0349	.9994	.0349	47°	.7314	.6820	1.0724
3°	.0523	.9986	.0524	48°	.7431	.6691	1.1106
4°	.0698	.9976	.0699	49°	.7547	.6561	1.1504
5°	.0872	.9962	.0875	50°	.7660	.6428	1.1918
6°	.1045	.9945	.1051	51°	.7771	.6293	1.2349
7°	.1219	.9925	.1228	52°	.7880	.6157	1.2799
8°	.1392	.9903	.1405	53°	.7986	.6018	1.3270
9°	.1564	.9877	.1584	54°	.8090	.5878	1.3764
10°	.1736	.9848	.1763	55°	.8192	.5736	1.4281
11°	.1908	.9816	.1944	56°	.8290	.5592	1.4826
12°	.2079	.9781	.2126	57°	.8387	.5446	1.5399
13°	.2250	.9744	.2309	58°	.8480	.5299	1.6003
14°	.2419	.9703	.2493	59°	.8572	.5150	1.6643
15°	.2588	.9659	.2679	60°	.8660	.5000	1.7321
16°	.2756	.9613	.2867	61°	.8746	.4848	1.8040
17°	.2924	.9563	.3057	62°	.8829	.4695	1.8807
18°	.3090	.9511	.3249	63°	.8910	.4540	1.9626
19°	.3256	.9455	.3443	64°	.8988	.4384	2.0503
20°	.3420	.9397	.3640	65°	.9063	.4226	2.1445
21°	.3584	.9336	.3839	66°	.9135	.4067	2.2460
22°	.3746	.9272	.4040	67°	.9205	.3907	2.3559
23°	.3907	.9205	.4245	68°	.9272	.3746	2.4751
24°	.4067	.9135	.4452	69°	.9336	.3584	2.6051
25°	.4226	.9063	.4663	70°	.9397	.3420	2.7475
26°	.4384	.8988	.4877	71°	.9455	.3256	2.9042
27°	.4540	.8910	.5095	72°	.9511	.3090	3.0777
28°	.4695	.8829	.5317	73°	.9563	.2924	3.2709
29°	.4848	.8746	.5543	74°	.9613	.2756	3.4874
30°	.5000	.8660	.5774	75°	.9659	.2588	3.7321
31°	.5150	.8572	.6009	76°	.9703	.2419	4.0108
32°	.5299	.8480	.6249	77°	.9744	.2250	4.3315
33°	.5446	.8387	.6494	78°	.9781	.2079	4.7046
34°	.5592	.8290	.6745	79°	.9816	.1908	5.1446
35°	.5736	.8192	.7002	80°	.9848	.1736	5.6713
36°	.5878	.8090	.7265	81°	.9877	.1564	6.3138
37°	.6018	.7986	.7536	82°	.9903	.1392	7.1154
38°	.6157	.7880	.7813	83°	.9925	.1219	8.1443
39°	.6293	.7771	.8098	84°	.9945	.1045	9.5144
40°	.6428	.7660	.8391	85°	.9962	.0872	11.4300
41°	.6561	.7547	.8693	86°	.9976	.0698	14.3010
42°	.6691	.7431	.9004	87°	.9986	.0523	19.0810
43°	.6820	.7314	.9325	88°	.9994	.0349	28.6360
44°	.6947	.7193	.9657	89°	.9998	.0175	57.2900
45°	.7071	.7071	1.0000				

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